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USERS' MANUAL FOR KALKER'S "EXACT" NONLINEAR CREEP THEORY

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INTERIM REPORT

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16. Abstract

The conversion of the computer program, "A Programme for Three-Dimensional Steady State Rolling" developed by Professor J. J. Kalker, from the original Algol language to Fortran is considered. This program determines the resultant creep forces and moment for steady state rolling of two bodies of equal or unequal linearly elastic material properties.

A related manual for Kalker's "Simplified Theory of Rolling Contact" is considered in the report "User's Manual for Kalker's Simplified Nonlinear Creep Theory," by James G. Goree and E. Harry Law, FRA/ORD-78/06 Contract DOT-OS-40018, December, 1977. The program considered in the present report concerns the same problem except for the extension to unequal materials. It is found that, for equal materials, the "Simplified Theory" gives approximately the same results as the exact solution in most cases and in those instances where some difference was noted, the simplified theory appears to be in better agreement with experimental results. In addition, the simplified theory reduces the computation time by a factor of approximately 50 to 100.

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Last, I would like to thank Professor J. J. Kalker of Delft who very graciously sent copies of his papers and computer programs, and gave his permission to include a reprint of one paper in this report. Although many checks have been made to verify the accuracy of the Fortran version of Professor Kalker's program, any errors in the conversion are mine and not Professor Kalker's.

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I. INTRODUCTION

A portion of the work supported by contract DOT-OS-40018, Freight Car Dynamics, concerned the conversion of two computer programs, obtained from Professor J. J. Kalker of Delft University, from the original Algol language to Fortran. In addition, detailed users' manuals were to be prepared for each program. The first program, the formulation of which is described by Kalker in [1], has been converted to Fortran and the users' manual is presented in [2]. The present users' manual covers the Fortran version of the computer program developed by Kalker in [3] and [4]. This program is a modification, by Kalker, of the original code described in [5].

Some duplication exists between the present manual and [2]. This is done both for completeness in the description of the problem, as both codes concern the same problem, and for ease of operation in that, where possible, the same nomenclature and input and output format is used in both programs.

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[1] J.J. Kalker, "Simplified Theory of Rolling Contact," Delft Progr. Rep., Series C: Mechanical and Aeronautical Engineering and Shipbuilding, 1 (1973), pp. 1-10. Reprint attached as Appendix B.

[2] J.G. Goree and E.H. Law, "Users' Manual for Kalker's Simplified Nonlinear Creep Theory," Interim Report, Contract DOT-OS-40018, FRA/ORD/-78/06. December, 1977.

[3] J.J. Kalker, "A Programme for Three-Dimensional Steady State Rolling. I Description of the Method." (1972), Unpublished.

[4] H. Goedings, "A Programme for Three-Dimensional Steady State Rolling. II Programme Description." (1972), Unpublished.

[5] J.J. Kalker, "On the Rolling Contact Between Two Elastic Bodies in the Presence of Dry Friction," Ph.D. Thesis, Delft University of Technology (1967).

Background

The forces and moments due to shear stresses in the contact area between wheel and rail play a major role in rail vehicle dynamics. These shear stresses arise, in part, due to relative linear and angular motions (lateral, longitudinal, and spin creepage) between the wheel and rail. Hobbs [6] presents a review of the analytical and experimental work concerned with the creep force/creepage phenomenon.

For many problems in rail vehicle dynamics a linear creep force/creepage relationship has been used. Typical of these are eigenvalue/eigenvector analyses of lateral stability, lateral forced response studies, and estimation of slip and flange contact boundaries for steady state curving. It is widely recognized that the best available linear creep law is that due to Kalker [1] and called the "linearized theory" (see equations (12) and (13) of [1]). Recently, however, more and more questions are being asked of rail vehicle dynamicists that require more sophisticated models of the wheel/rail interaction process. Factors that should be considered in these models are: (1) the nonlinear wheel/rail geometric constraint functions arising from curved or worn wheel and rail profiles; and, (2) the effects of adhesion limits on the creep force/creepage relationship (i.e. a nonlinear creep law).

Attempts have been made to formulate a nonlinear creep law.

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[6] A.E.W. Hobbs, "A Survey of Creep", DYN/52, April 1967, British Railways Research Dept., Derby, England.

Johnson's theory [7,8] has been confirmed by laboratory experiments but does not account for spin creepage*. Unfortunately, the effects of spin creepage are expected to predominate for contact areas in the wheel flange region - precisely the situation where a nonlinear creep law is needed. The Levi-Chartet creep law [9,10] used by some researchers is empirically based and does not account for spin creepage.

Professor Kalker of Delft University has formulated two nonlinear creep laws that incorporate the effects of spin creepage and that have been found to compare well with results of laboratory experiments. These two creep laws are generally referred to as the "simplified theory of rolling contact" [1] and the "exact solution for rolling contact" [3], [5]. The differences in the solutions lie in two simplifying assumptions made in [1] concerning the tangential displacement-stress relations and the normal stress distribution on the contact surface. These assumptions shorten the computation time required by a factor of approximately 50 to 100 for the simplified theory. The exact solution is valid for unequal materials while equal material properties must be assumed in the simplified theory.

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[7] K.L. Johnson, "Adhesion", Proc. Inst. Mech. Engrs., Vol. 178, part 3E (1964), pp. 208, 209.

[8] P.J. Vermeulen and K.L. Johnson, "Contact of Nonspherical Elastic Bodies Transmitting Tangential Forces," J. Appl. Mechanics, Vol. 31 (1964), pp. 338-340.

* Spin creepage is the nondimensional relative angular velocity between wheel and rail in the contact zone.

[9] R. Levi, "Le roulement avec glissement", Compt. rend. Acad. Science 199, (1934), pp. 119-120.

[10] A. Chartet, "Proprietes generales des contacts de roulement. Theorie des similitudes." Compt. rend. Acad. Science 225, (1947), pp. 986-988.

Some of the investigations being conducted under contract DOT-OS-40018, Freight Car Dynamics, deal with developing models for the lateral dynamic response of North American freight cars during curve entry and negotiation. These models will be used to predict vehicle response and wheel/rail forces during hard curving where severe flange contact is anticipated. Consequently, it is expected that creep forces may approach the limits of adhesion and a nonlinear creep law will be required for accurate modeling.

The object of the work reported in this Users' Manual was to convert the Algol program developed by Professor Kalker for the "three-dimensional steady state theory of rolling contact" to Fortran and to check the resulting program by direct comparison with the results calculated by the original Algol program and with available experimental results. It is anticipated that a Fortran version of this computer program and the simplified theory of [2], will prove quite valuable to rail vehicle dynamics researchers in the United States where most scientific programs are written in Fortran.

Summary of Users' Manual

The problem analysed in [3] and considered in the computer code is for steady rolling contact of two elastic bodies of equal or unequal linearly elastic material properties and having both longitudinal and lateral creepage and spin about an axis normal to the contact surface. The appropriate geometry is given in Figure 1.

The problem may be stated as follows. Given two bodies of known elastic properties, dimensions, normal force, rolling velocity, creepage and spin, determine the resultant creep forces tangent to the contact surface. The region of slip within the contact surface is also determined. In obtaining a solution, the static Hertzian contact problem is first solved (see [5] page 55, or [11] page 414) to determine the dimensions of the contact ellipse, a and b. The resultant creep forces and moment, F_x, F_y, and M_z are then determined knowing the parameters a, b, N, G, ν, κ, μ, ν_x, ν_y, and ϕ where:

FX = F_x = longitudinal creep force (in the direction of rolling)

FY = F_y = lateral creep force

MZ = M_z = spin creep moment about normal to contact surface

A1 = a = semi-axis of contact ellipse in longitudinal direction

B1 = b = semi-axis of contact ellipse in lateral direction

N = resultant normal load on the contact region

G = combined shear modulus, $1/G = 1/2(1/G^+ + 1/G^-)$

NU = ν = combined Poisson's ratio, $\nu/G = 1/2(\nu^+/G^+ + \nu^-/G^-)$

Kappa = κ = elastic difference parameter, $= G/4[(1-2\nu^+)/G^+-(1-2\nu^-)/G^-]$

MU = μ = coefficient of friction

UY,UY = ν_x, ν_y = longitudinal and lateral creepage

PH = ϕ = spin creepage

The significant differences in the solutions presented in [1] and [3] lie in two simplifying assumptions concerning the tangential displacement - stress relations and the normal stress distribution on the contact surface. These two assumptions considerably reduce the complexities in obtaining a numerical solution and shorten the computation time

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[11] S.P. Timoshenko and J.N. Goodier, Theory of Elasticity, 3rd Ed., McGraw-Hill Book Company (1970).

by a factor of approximately 50 to 100.

The first assumption regarding the tangential displacement-stress relation developed in [1] is,

$$u^+(x,y) - u^-(x,y) = u(x,y) = S_x X = - S_x \tau_{xz}$$
$$v^+(x,y) - v^-(x,y) = v(x,y) = S_y Y = - S_y \tau_{yz} \qquad \text{equation (9), [1]}$$

where u(x,y) and v(x,y) are the tangential displacement differences in the longitudinal and lateral directions and τ_{xz} and τ_{yz} are the shear stresses. The "exact" relationships for the tangential displacements as given in [3] and [5] are

$$u(x,y) = - \frac{1}{\pi G} \iint_A \left\{ \tau_{xz}(p,q) \left[\frac{1-\nu}{R} + \frac{\nu(p-x)^2}{R^3}\right] + \tau_{yz}(p,q) \frac{\nu(p-x)(q-y)}{R^3} + \kappa\sigma_z(p,q) \frac{p-x}{R^2} \right\} dp\, dq$$

$$= \sum_{m=o}^{M} \sum_{n=o}^{M-m} a_{mn}\, x^m y^n \quad \text{, and}$$

$$v(x,y) = - \frac{1}{\pi G} \iint_A \left\{ \nu\tau_{xz}(p,q) \frac{(p-x)(q-y)}{R^3} + \tau_{yz}(p,q) \left[\frac{1}{R} - \frac{\nu(p-x)^2}{R^3}\right] + \kappa\sigma_z(p,q) \frac{q-y}{R^2} \right\} dp\, dq$$

$$= \sum_{m=o}^{M} \sum_{n=o}^{M-m} b_{mn}\, x^m y^n$$

where $R = \sqrt{(x-p)^2 + (y-q)^2}$, and A is the contact area.
The two elastic constants S_x and S_y of [1] are determined explicity in terms of the elastic properties G and ν, the contact ellipse dimensions a and b and the creepage and spin coefficients C_{ij} (see equations (13) and (41) - (47) of [1].)

The method of determination of the constants a_{mn} and b_{mn} in the "exact" solution is much more complicated than that used to determine S_x and S_y in [1] and is the significant difference in the solutions. The coupling of the shear stresses and the normal stress σ_z is apparent in the above "exact" relations. However if the materials are equal, and therefore $\kappa = 0$, the normal stress contribution does vanish. For unequal materials this contribution may be significant and represents the main difficulty in developing a simplified theory for unequal materials.

Both theories also may be used to investigate the effects of a very thin elastic layer covering the bodies and having a tangential displacement-stress relation as given by equation (45) of [1].

$$u_\ell = L_x X = - L_x \tau_{xz}, \text{ and}$$

$$v_\ell = L_y Y = - L_y \tau_{yz},$$

where L_x and L_y are the inverse stiffnesses of the layer. If no layer is present one then takes $L_x = L_y = 0$.

The effect of changes in L_x and L_y on the resulting solution has not been investigated; however, some observations should be noted. First, the layer is assumed to be so thin that its presence does not influence the determination of the contact ellipse dimensions or the pressure distribution. That is, a and b are still computed from the static Hertz solution in terms of G, ν, and N. The effect of a finite thickness work-hardened layer could not then be accounted for by including L_x and L_y. Further, it seems to the writer that if the effect of a contaminated rail is desired, it is more directly accounted for by an appropriate change in the coefficient of friction. The utility of modifying the elastic properties by adding L_x and L_y is not clear to the writer at this time.

The additional simplification made in the combined creepage and spin solution of [1] is that the normal stress distribution over the contact region is assumed to be of the form given by equation (14.III) of [1] rather than the Hertz stress distribution. It should be noted that the Hertzian distribution is used in both cases, [2] and the present code, to determine the contact region dimensions a and b.

In running test problems with the two programs derived from, [1] and [3], some important observations have been made. In comparing the solutions with experimental results for equal materials, as shown in Figure 2, the agreement is equally as good using the "Simplified Theory" [1] as the "Exact Theory" [3]. In fact, for small or large values of A/B the "Simplified Theory" frequently gives better results, as the "Exact Theory" often experiences numerical divergence difficulties for extreme values of A/B. In no instance was a significant improvement noted with the "Exact Theory". In view of the time savings on the order of 50 to 1 the "Exact Theory" appears to have utility primarily in verifying the "Simplified Theory". For unequal materials the "Exact Theory" must be used, however the solution time is considerably increased, as the normal stress is now coupled into the tangential displacements and convergence is more difficult. An indication of run times for specific examples is given in the next section.

Numerous changes were made in the computer code in order to make the program more convenient to use. The Algol version was, however, fundamentally correct and numerous checks were made to insure that the Fortran and Algol codes gave the same results. The use of the Fortran code is considered in the next sections.

II. DESCRIPTION OF COMPUTER CODE FOR THE "EXACT" SOLUTION

A. PURPOSE

This program and associated subroutines computes the lateral and longitudinal creep forces and the spin creep moment acting between two elastic bodies in steady state rolling contact. The bodies are of equal or unequal linearly elastic material properties and have longitudinal and lateral creepage and spin creepage about an axis normal to the contact region. Kalker's theory of three-dimensional steady state rolling contact [3], [4]is the basis of the program.

B. PROGRAM DESCRIPTION

1) Usage: The program consists of a main program and two subroutines.

   The main program, MAIN, coordinates the input, determines the region of slip or adhesion within the contact zone, and outputs the results. Subroutine CØNST determines the normalized modulus GS by linear and quadratic interpolation from Kalker's table [5].

2) Subroutines Required:

   SUBROUTINE SIGN (X) If the function X is negative, zero or positive the subroutine returns -1.0, 0.0, +1.0, respectively.

   SUBROUTINE CØNST (A, B, NU, GS) determines the normalized modulus, GS, by linear and quadratic interpotation from Kalker's table, [5]. These values are used in MAIN.

3) Description of Input Parameters:

   NV1 NV1 is an integer denoting the number of complete problems to be solved.

A,B	$A = a/c$, $B = b/c$, where a and b are the actual contact dimensions determined from the static Hertz solution and $c = \sqrt{ab}$ is the normalized unit of length. a is the longitudinal and b is the lateral semi-axis of the contact ellipse.
NU	NU = ν = Poisson's ratio.
LXN, LYN	LXN = $L_x \rho N/c^4$, LYN = $L_y \rho N/c^4$. Inverse stiffnesses of an elastic layer covering the bodies. N = resultant normal force and $1/\rho = 1/4\ (1/R_1^+ + 1/R_1^- + 1/R_2^+ + 1/R_2^-)$ with R_1^+, R_1^-, R_2^+, R_2^- being the principal radii of curvature of the two elastic bodies. See equation (45), [1]. For no layer, take LXN = LYN = 0.
KAPPA	KAPPA = Elastic difference parameter.
N1, M1	Lattice points in the normalized contact region, see Figure 1. Accuracy increases with increasing values of N1, M1. Maximum values N1, M1 = 8. Typical values: A/B = 10.0, N1 = 8, M1 = 6, A/B = 0.1, N1 = 6, M1 = 8, A/B = 1.0, N1 = M1 = 6.
NS	To print all output including stresses and displacements on the contact region take NS = 1. To suppress all output except the resultant forces or moment take NS = 2.
NV2	NV2 is the integral number of sets of UXN, UYN, PHN to be considered.

UXN, UYN   UXN = $\nu_x \rho/\mu c$, UYN = $\nu_y \rho/\mu c$ where ν_x, ν_y are the longitudinal and lateral creepages, μ = coefficient of friction.

PHN   PHN = $\phi\rho/\mu$ where ϕ is the spin creepage.

4) Input Format:

A sample deck set up is listed in Appendix A of this manual. The program requires contact region dimensions, elastic properties, wheel/rail creepages and program control information. The following format is for NV1 = 1. If NV1 > 1, there would be NV1 sets of the group of cards after the first card.

Card Number	Input Data
1	NV1 = Integer. Program solves NV1 complete problems, Typical card: 1
2	A, B, NU, LXN, LYN, KAPPA Typical card: 2.5980 0.3849 0.28 0.00 0.00 0.00
3	N1, M1, NS Typical card: 6 6 1
4	NV2 = Integer. Program solves NV2 problems for different values of creepage and spin given on NV2 cards starting with 5. Typical card: 1
5 to NV2	UXN, UYN, PHN Typical card: 0.0 -1.4 0.8

Note: The input is free format with a space needed between each input parameter.

5) <u>Description of Other Parameters in Program:</u>

GS $GS = Gc^3/\rho N$ where G = shear modulus. GS may also be computed from $GS = 3(1-\nu)\ \tilde{E}/(4\pi\sqrt{g})$ where $\tilde{E}$ = complete elliptic integral of the second kind, see [5] page 58, and g = axial ratio of the contact ellipse = min (a/b, b/a). GS is determined within the computer program in terms of A, B and NU.

MU MU = μ = coefficient of friction. All variables are normalized so that μ does not appear explicitly.

6) <u>Output:</u> NV2 sub-cases of NV1 cases are calculated. For each of the NV1 cases, the input parameters A, B, NU, LXN, LYN, KAPPA are printed. The constants N1, M1, NS and the normalized shear modulus, GS, are also printed. For each of the NV1 cases, there will be NV2 sets of output corresponding to the NV2 sets of normalized creepages and spin, UXN, UYN, and PHN. For each of the NV2 cases, the inputs UXN, UYN, and PHN are printed out together with the computed values of the normalized longitudinal and lateral creep forces, FXN and FYN, and the computed value of the spin creep moment, MZN. If NS = 2, the output is as described above. If NS = 1, the normalized coordinate points X, Y over the contact region and the values of the stresses (TX, TY, TZH) and slip components (VX, VY) are given at each point.

The Fortran names used in the program output are the following, and are listed in the order of printing.

UXN, UYN, PHN Repeated program input variables.

Two different error messages may be printed after the above input variables. The first occurs when the numerical procedure is unable to satisfy the error bounds built into the program. For this case the statement PROCESS INTERUPTED, RESULTS MAY NOT BE SIGNIFICANT is printed and the calculated results are printed. The second error message occurs when a matrix within the program becomes singular and no results can be calculated. For this case the statement SINGULAR MATRIX, NO RESULTS is printed.

X, Y	$X = x/c$, $Y = y/c$. $-A \le X \le A$, $-B \le Y \le B$. Normalized coordinates where x and y are longitudinal and lateral distances from the center of the contact ellipse.
TX, TY	Normalized shear stresses $TX, TY = -\tau_{xz}c^3/\rho N, -\tau_{yz}c^3/\rho N,$ $\sqrt{TX^2 + TY^2} = TZH$ for no slip, $\sqrt{TX^2 + TY^2} = TZH$ for slip.
TZH	$TZH = 3/(2\pi) \sqrt{1-(X/A)^2 - (Y/B)^2}$ = Normalized Hertzian stress on the contact region.
VX, VY	Normalized relative slip components. $VX, VY = v_x\rho/(V\mu c), v_y\rho/(V\mu c)$ where V is the rolling velocity and v_x and v_y are the longitudinal and lateral components of the relative slip velocity.
FXN, FYN	$FXN = F_x/\mu N$, $FYN = F_y/\mu N$. Normalized resultant longitudinal and lateral forces. Computed.
MZN	$MZN = M_z c/\mu N$. Normalized resultant moment. Computed.

7) Summary of User Requirements and Recommendations:

All input data is on cards in free format as shown. As A and B are normalized, the product of A and B must be unity. LXN and LYN are taken as zero if no elastic layer is to be considered. Maximum values for N1 and M1 are 8. Accuracy increases with increasing values of N1 and M1. Typical values are:

A/B = 10.0 N1 = 8, M1 = 6

A/B = 1.0 N1 = M1 = 6

A/B = 0.1 N1 = 6, M1 = 8

C. PROGRAM LISTINGS WITH EXAMPLE INPUT AND OUTPUT

A listing of the program for a sample problem with input and output is given in Appendix A.

D. SAMPLE PROBLEM

The sample problem of Appendix A is for the input listed below. The calculations were performed on an IBM-370/3165-II computer.

A = 2.598, B = 0.3849, NU = 0.28, LXN = 0.0, LYN = 0.0, KAPPA = 0.0

N1 = 6, M1 = 6, NS = 1

UXN = 0, UYN = -1.4, PHN = 0.8

III. DISCUSSION OF RESULTS OF USE OF PROGRAM

This Fortran computer program has been run on the Clemson University IBM-370/3165-II computer. The Clemson computer is equipped with a CDC speed-up processer and is approximately three times as fast as a standard IBM 370-165. Typical computation time for a complete solution with full output on the contact region was about 60 seconds for A = 2.598, B = 0.3849, N1 = 6, M1 = 6, KAPPA = 0.0. Running sequential problems and therefore reducing the compile time for each problem reduced the above times to approximately 40 seconds. The same problem with KAPPA = 0.2 required 255 seconds.

Many particular examples have been worked using this code and comparisons have been made with the results of [2] and [3]. These comparisons have shown excellent agreement while the numerical solution technique of [5] has convergence difficulties in this range. The present numerical solution technique seems to converge much better than that used in [5], although still not as smoothly as the simplified theory of [2].

Of perhaps more interest is the comparison of the theory with experimental studies. Surprisingly good agreement is demonstrated in Figure 2 of this text where the results are compared with the simplified theory [2], and with the experimental results of Gilchrist and Brickle [12]. Only the case of A/B = 6.75 is shown on the figure; however, equally good agreement was found for A/B = 1.11 and A/B = 10.3.

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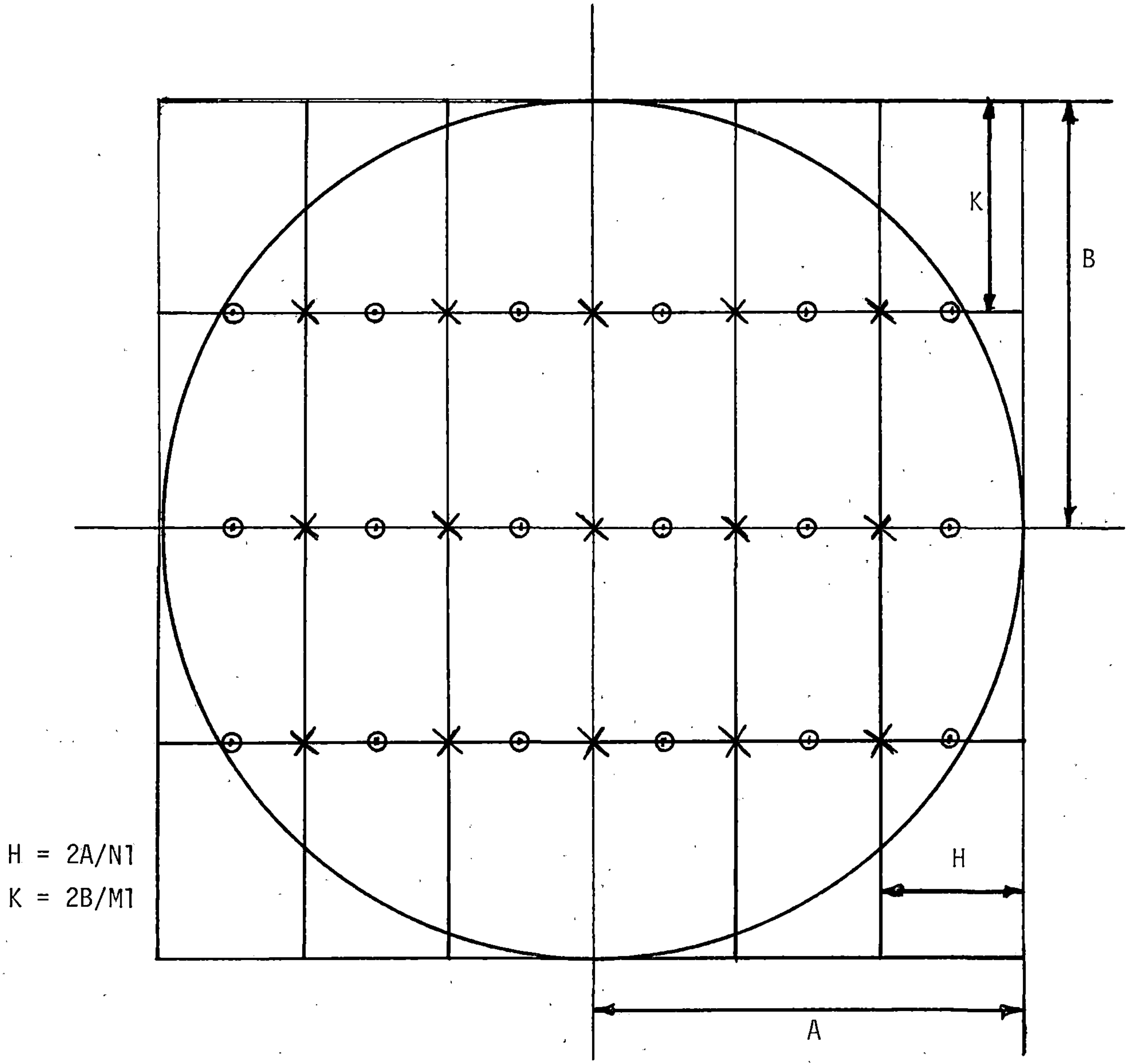
[12] A.O. Gilchrist and B.V. Brickle, "A Re-examination of the Proneness to Derailment of a Railway Wheel-Set." J. Mech. Engr. Sci., Vol. 18, No. 3, (1976), pp. 131-141.

As in the simplified theory the resultant creep forces and moment are not strongly dependent on the number of lattice points N1 and M1. The accurate determination of the slip, no-slip zones within the contact region is more dependent on these parameters. The increase in computation time with increase in the number of lattice points N1 and M1 is more significant in the present code than in [2]. For example, as stated above for A = 2.598, B = 0.3849, N1 = M1 = 6, Kappa = 0.0 the computation time was approximately 60 seconds. Increasing N1 and M1 to N1 = M1 = 8 increased the computation time to 4 minutes and 45 seconds. The values of the resultant force in the lateral direction as shown in Figure 2 was FYN = -0.411 for N1 = M1 = 6 and FYN = -0.455 for N1 = M1 = 8. The second value of FYN = -0.455 is seen to be closer to the experimental results of [12].



N = number of traction points indicated by X

M = number of slip points indicated by 0
(the slip points lie midway between the traction points)

In the above figure A = B = 1, N1 = 6, M1 = 4, N = 15, M = 18

FIGURE 1. Normalized Contact Region (A*B=1.0)

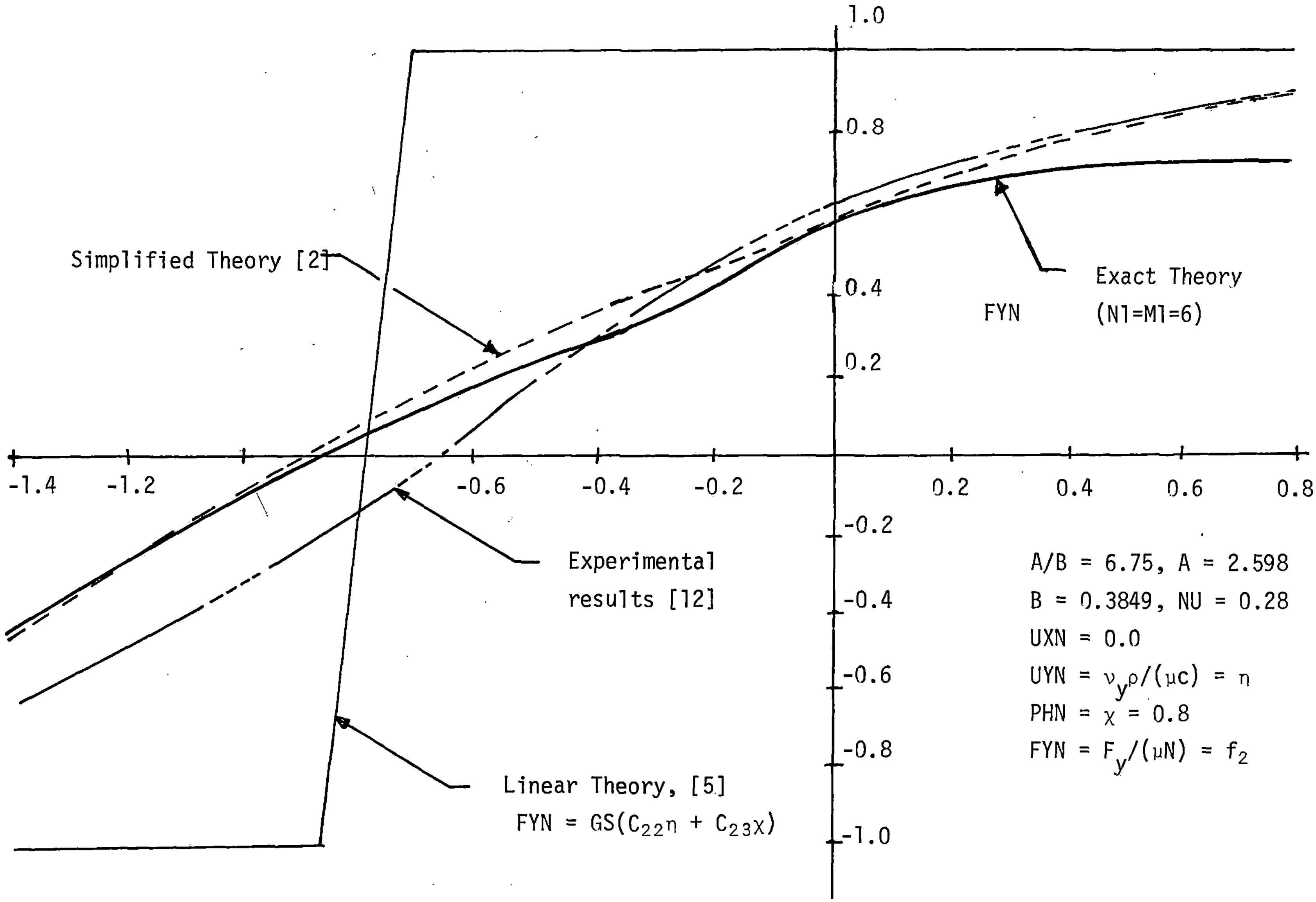


FIGURE 2. Comparison of Kalker's "Exact" Theory with the Simplified Theory and with the Experimental Results of [12]. (See Figure 7, [12]).

APPENDIX A

LISTING AND TEST PROBLEM

(FORTRAN IV G1 RELEASE 2.0)

This program is referred to as
PROGRAM WISK-SRT by Kalker

MAIN

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C     SEE "ON THE ROLLING CONTACT OF TWO ELASTIC BODIES IN THE
C     PRESENCE OF DRY FRICTION," BY J.J. KALKER, PH.D THESIS,
C     DELFT UNIVERSITY (1967). THE PROGRAM IS AN IMPROVEMENT ON
C     METHOD DESCRIBED IN THE ABOVE THESIS AND IS PRESENTED IN
C     PAPER "A PROGRAMME FOR THE THREE-DIMENSIONAL STEADY STATE
C     ROLLING. I, DESCRIPTION," BY J.J. KALKER AND "II,
C     PROGRAMME DESCRIPTION," BY H. GOEDINGS (1972).
C     UNPUBLISHED, OBTAINED FROM PROFESSOR KALKER IN PRIVATE
C     COMMUNICATION.
C
C
C     THE INPUT IS DESCRIBED IN THE FOLLOWING SECTION
C
C
C     DATA CARD #1      NV1
C                       TYPICAL CARD: 1
C
C                       NV1 (SOLVES NV1 COMPLETE PROBLEMS), INTEGER
C
C     DATA CARD #2      A,B,NU,LXN,LYN,KAPPA
C                       TYPICAL CARD: 2.5980 0.3849 0.28 0.00 0.00 0.00
C
C                       ( A AND B ARE THE NORMALIZED CONTACT ELLIPSE
C                       DIMENSIONS, WHERE IF A1 AND B1 ARE THE ACTUAL
C                       DIMENSIONS THEN A=A1/SQRT(A1*B1) AND
C                       B=B1/SQRT(A1*B1). NOTE A/B >= 0.1
C                       NU IS THE COMBINED POISSON'S RATIO. WHERE
C                       NU=G/2*(NU+/G+ + NU-/G-), WHERE THE + AND
C                       - SIGNS REFER TO POISSON'S RATIO AND THE
C                       SHEAR MODULUS FOR THE LOWER AND UPPER
C                       REGION RESPECTIVELY. THE CONSTANT G IS THE
C                       COMBINED MODULUS, 1/G=1/2*(1/G+ + 1/G-).
C                       THE VALUES OF A,B,NU PROVIDE THE NECESSARY
C                       INFORMATION NEEDED TO COMPUTE (INTERNALLY) FROM
C                       KALKER'S TABLES AND ASYMPTOTIC EXPANSIONS, ( SEE
C                       SUBROUTINE CONST ), THE NORMALIZED MODULUS,GS
C                       THE CONSTANT GS=G*(C***3)/(RHO*N),WHERE C=
C                       SQRT(A1*B1), 1/RHO=1/4*(1/R1+ + 1/R1- + 1/R2+ +
C                       1/R2-),AND N=RESULTANT NORMAL FORCE.
C                       LXN AND LYN ARE NORMALIZED INVERSE STIFFNESSES
C                       OF A THIN ELASTIC LAYER COVERING THE SURFACE.
C                       LXN=LX*RHO*N/C**4, LYN=LY*RHO*N/C**4.
C                       FOR NO LAYER TAKE LXN=LYN=0.0
C                       KAPPA IS THE ELASTIC DIFFERENCE PARAMETER
C                       KAPPA=G/4*((1-2*NU+)/G+-(1-2*NU-)/G-).
C
C     DATA CARD #3      N1,M1,NS
C                       TYPICAL CARD: 12,6,2
C
C                       N1,M1 (LATTICE POINTS IN CONTACT REGION,
C                       NS (TO PRINT OUTPUT ON THE CONTACT REGION,NS=1,
C                       TO SUPPRESS ALL OUTPUT EXCEPT THE RESULTANT
C                       FORCES AND MOMENT , TAKE NS=2), INTEGER
C                       NOTE: FXN=FX/(MU*N), FYN=FY/(MU*N),
C                       MZN=MZ*C/(MU*N),WHERE N IS THE RESULTANT NORMAL
C                       FORCE AND MU IS THE COEFFICIENT OF FRICTION
C
C     DATA CARD #4      NV2
```

```
C                     TYPICAL CARD: 1
C
C                     SOLVES NV2 PROBLEMS FOR DISTINCT VALUES OF
C                     CREEPAGE AND SPIN GIVEN ON NV2 CARDS 5), INTEGER
C
C     DATA CARD #5    UXN,UYN,PHN
C                     TYPICAL CARD: 0.0 2.0 0.4
C
C                     UXN AND UYN ARE NORMALIZED CREEPAGES, PHN
C                     IS THE NORMALIZED SPIN), REAL
C                     UXN=UX*RHO/(MU*C), UYN=UY*RHO/(MU*C),
C                     PHN=PH*RHO/MU
C
C
C               ***** NOTE: ALL VARIABLES HAVE BEEN NORMALIZED SUCH
C               ***** THAT THE COEFFICIENT OF FRICTION, MU, DOES NOT
C               ***** APPEAR EXPLICITLY.
C
C
      DIMENSION XS(38),XT(400),YT(400),ZT(400),XU(400),YU(400),ZU(400)
      DIMENSION RZT(120,1),RZU(120,1),F1(60,60),F2(60,60),F3(60,60)
      DIMENSION S(60),FACC(1,120),FDACC(120,120),U1(1,120),U2(120,120)
      REAL KAPPA
      INTEGER C(38),C1(18),C2(18),TMA(200)
      INTEGER PMA,QMA,FAC1,FAC2
      DIMENSION XTU(60),YTU(60)
      DIMENSION ARR(120,120),T(120,1),U(120,1),RT(120),TT(120),P(120)
      DIMENSION RU(120)
      REAL MU,KPG,K1,K2,K3,MZS,LXN,LYN
      INTEGER FAC1P,FAC2P,WP
      REAL MZ
      EXTERNAL SIGN
      DATA C1/1,0,-1,-2,0,2,1,0,-1/,C2/1,-2,1,-2,4,-2,1,-2,1/
      REAL K
      REAL NU
      DATA PI/3.14159/
      READ(1,*)NV1
      DO 999 II1=1,NV1
C
      READ(1,*)A,B,NU,LXN,LYN,KAPPA
      READ(1,*)N1,M1,NS
C
      SX=LXN
      SY=LYN
      IF(A/B.LT.0.1) GO TO 998
C
C
C                    SUBROUTINE CONST COMPUTES THE NORMALIZED MODULUS
C                    FROM KALKER'S TABLES AND ASYMPTOTIC EXPANSIONS.
C                    VALID FOR A/B EQUAL TO OR GREATER THAN 0.1 .
C
C
      CALL CONST(A,B,NU,GS)
      G=GS
      SIGMA=NU
      F00=3.0/(2.0*PI)
      MU=1.0
      H=    A/FLOAT(N1)
```

```
      K=(2.0*B)/FLOAT(M1)
      WRITE(3,968)
      N=0
      M=0
      AA=A*A
      Y=B
      MM=M1/2
      L3=0
      L4=0
      DO 100 I=1,MM
      L1=0
      L2=0
      Y=B-I*K
      YB=(Y*Y)/(B*B)
      XS(M1-I)=-A*SQRT(1.0-YB)
      XS(I)=XS(M1-I)
      X=-XS(I)/(2*H)
      J=X
      IF((-2.*J*H-XS(I))/H.LE.0.02) J=J-1
      L=2*J+1
      X=-L*H
      IF(X.LT.XS(I))GO TO 50
      M=M+1
      L2=1
      XU(M)=X
      YU(L+M)=Y
      YU(M)=Y
      ZU(L+M)=F00*SQRT(1.0-(X*X)/AA-YB)
      ZU(M)=ZU(L+M)
      XU(L+M)=-X
50    N=N+1
      X=X+H
      IF(X.LE.-.1*H) GO TO 200
      X=0.0
      XT(N)=0.0
      YT(N)=Y
      ZT(N)=F00*SQRT(1.0-YB)
      GO TO 60
200   L=N+2*(J-L1)
      ZT(L)=F00*SQRT(1.0-(X*X)/AA-YB)
      ZT(N)=ZT(L)
      XT(N)=X
      XT(L)=-X
      YT(L)=Y
      YT(N)=Y
      X=X+H
      M=M+1
      L=M+2*(J-L1)-1
      XU(M)=X
      XU(L)=-X
      YU(M)=Y
      YU(L)=Y
      ZU(M)=F00*SQRT(1.0-(X*X)/AA-YB)
      ZU(L)=ZU(M)
      L1=L1+1
      GO TO 50
60    N=N+J
      M=M+J+L2
```

```
      C(2*I-1)=N-L3
      C(2*I)=M-L4
      L3=N
      L4=M
100   CONTINUE
      NN=N+1
      MM=M+1
      L=0
      N=2*N-N1+1
      M=2*M-N1
      I=N
301   L=L+1
      XT(I)=-XT(L)
      YT(I)=-YT(L)
      ZT(I)=ZT(L)
      I=I-1
      IF(I.GE.NN) GO TO 301
      L=0
      I=M
401   L=L+1
      XU(I)=-XU(L)
      YU(I)=-YJ(L)
      ZU(I)=ZU(L)
      I=I-1
      IF(I.GE.MM) GO TO 401
      L=3
      H=2.*H
      ISTART=M1/2+1
      IEND=M1-1
      DO 500 I=ISTART,IEND
      IF(ISTART.GT.IEND)GO TO 500
      C(2*I-1)=C(M1-L)
      C(2*I)=C(M1-L+1)
500   L=L+2
      WRITE(3,901)
901   FORMAT(1H1)
      WRITE(3,969)
      WRITE(3,970)A,B,NU,LXN,LYN,KAPPA
      WRITE(3,972)N1,M1,NS
      WRITE(3,973)GS,N,M
C
C
      MAX=M
      IF(N.GT.M) MAX=N
C
C
C     IN-LINE MRZ
C     F4=F1,    F5=F2,    RZ=RZU,    CMRZ=0.5
      MN=M
      DO 8110 I=1,M
      XTU(I)=XU(I)
      RZU(2*I-1,1)=0.0
      RZU(2*I,1)=0.0
8110  YTU(I)=YU(I)
8130  DO 8140 I=1,MN
      DO 8140 J=1,N
      L=0
      P4=0.0
```

```
      P5=0.0
      Q4=0.0
      Q5=0.0
      X1=XT(J)-XTU(I)
      Y1=YT(J)-YTU(I)
      X=X1-H
 8131 Y=Y1-K
 8132 L=L+1
      T1=ALOG(X*X+Y*Y+H*1.E-10)
      T2=X*ATAN(Y/(X+H*1.E-10))
      T3=Y*ATAN(X/(Y+K*1.E-10))
      P4=P4+C1(L)*(.5*Y*T1+T2)
      P5=P5+C1(L)*(.5*X*T1+T3)
      Q4=Q4+C2(L)*((X*X+Y*Y)*(T1-1)-(.5*Y*T1+T2)*Y1*4.0)
      Q5=Q5+C2(L)*(X*Y+Y*T3-X*T2-(.5*X*T1+T3)*Y1*2.0)
      Y=Y+K
      IF(Y.LE.Y1+K+.5*K) GO TO 8132
      X=X+H
      IF(X.LE.X1+H+.5*H) GO TO 8131
      F1(I,J)=P4/H+Q4/H/K/4.0
      F2(I,J)=P5/H+Q5/H/K/2.0
 8140 CONTINUE
      LEND=2*MN-1
      DO 8150 I=1,LEND,2
      DO 8150 J=1,N
      RZU(I,1)=RZU(I,1)+F1((I+1)/2,J)*ZT(J)
 8150 RZU(I+1,1)=RZU(I+1,1)+F2((I+1)/2,J)*ZT(J)
C     IN-LINE MRZ
C     F4=F1,   F5=F2,   RZ=RZT,   CMRZ=1.E-5
 8220 MN=N
      DO 8225 J=1,N
      XTU(J)=XT(J)
      RZT(2*J-1,1)=0
      RZT(2*J,1)=0.0
 8225 YTU(J)=YT(J)
 8230 DO 8240 I=1,MN
      DO 8240 J=1,N
      L=0
      P4=0.0
      P5=0.0
      Q4=0.0
      Q5=0.0
      X1=XT(J)-XTU(I)
      Y1=YT(J)-YTU(I)
      X=X1-H
 8231 Y=Y1-K
 8232 L=L+1
      T1=ALOG(X*X+Y*Y+H*1.E-10)
      T2=X*ATAN(Y/(X+H*1.E-10))
      T3=Y*ATAN(X/(Y+K*1.E-10))
      P4=P4+C1(L)*(.5*Y*T1+T2)
      P5=P5+C1(L)*(.5*X*T1+T3)
      Q4=Q4+C2(L)*((X*X+Y*Y)*(T1-1)-(.5*Y*T1+T2)*Y1*4.0)
      Q5=Q5+C2(L)*(X*Y+Y*T3-X*T2-(.5*X*T1+T3)*Y1*2.0)
      Y=Y+K
      IF(Y.LE.Y1+K+.5*K) GO TO 8232
      X=X+H
      IF(X.LE.X1+H+.5*H) GO TO 8231
```

```
      F1(I,J)=P4/H+Q4/H/K/4.0
      F2(I,J)=P5/H+Q5/H/K/2.0
8240  CONTINUE
      LEND=2*MN-1
      DO 8250 I=1,LEND,2
      DO 8250 J=1,N
      RZT(I,1)=RZT(I,1)+F1((I+1)/2,J)*ZT(J)
8250  RZT(I+1,1)=RZT(I+1,1)+F2((I+1)/2,J)*ZT(J)
      DO 83100 I=1,M
      DO 83100 J=1,N
      L=0
      P1=0
      P2=0
      P3=0
      Q1=0
      Q2=0
      Q3=0
      X1=XT(J)-XU(I)
      Y1=YT(J)-YU(I)
      Y1END=Y1+K+.5*K
      X1END=X1+H+.5*H
      X=X1-H
8310  Y=Y1-K
8320  L=L+1
      R=ABS(X/(Y+K*1.E-10))
      SS=ABS(Y/(X+H*1.E-10))
      T1=SIGN(X)*Y*ALOG(R+SQRT(1.+R*R))
      T3=SIGN(Y)*X*ALOG(SS+SQRT(1.+SS*SS))
      T2=SQRT(X*X+Y*Y+H*K*1.E-20)
      P1=P1+C1(L)*(T1+T3)
      P2=P2-C1(L)*T2
      P3=P3+C1(L)*T1
      Q1=Q1+C2(L)*(Y*T1+X*T2-(T1+T3)*Y1*2.)
      Q2=Q2+C2(L)*(X*T3-Y*T2+T2*Y1*2.)
      Q3=Q3+C2(L)*(Y*T1-X*T2-T1*Y1*2.)
      Y=Y+K
      IF(Y.LE.Y1END) GO TO 8320
      X=X+H
      IF(X.LE.X1END) GO TO 8310
      F1(I,J)=P1/H+Q1/H/K/2.
      F2(I,J)=P2/H+Q2/H/K/2.
      F3(I,J)=P3/H+Q3/H/K/2.
83100 CONTINUE
C
C
      READ(1,*)NV2
      WRITE(3,974) NV2
      DO 997 L2K=1,NV2
      READ(1,*,END=9999)UXN,UYN,PHN
      UX=UXN
      UY=UYN
      PHI=PHN
      WRITE(3,975)UXN,UYN,PHN
C
      L3=0
      L4=N
      L5=M
      IF(ABS(UX).LT.1.E-8.AND.ABS(KAPPA).LT.1.E-8) L3=1
```

```
      IF(ABS(UY).LT.1.E-8.AND.ABS(PHI).LT.1.E-8) L3=2
      IF(L3.EQ.0)GO TO 8410
C
C     IN-LINE KAPAF
      N=0
      M=0
      J=M1/2
      DO 8420 I=1,J
      M=M+C(2*I)
      N=N+C(2*I-1)
8420  CONTINUE
8410  CONTINUE
C
C
C     IN-LINE MA
      PMA=2*M-1
      QMA=2*N-1
      PIG=PI*G
      IF(L3)8520,8510,8520
8510  DO 8515 J=1,QMA,2
      DO 8515 I=1,PMA,2
      I1=(I+1)/2
      J1=(J+1)/2
      ARR(I,J)=((1.-SIGMA)*F1(I1,J1)+SIGMA*F3(I1,J1))/PIG
      ARR(I,J+1)=(SIGMA*F2(I1,J1))/PIG
      ARR(I+1,J)=ARR(I,J+1)
8515  ARR(I+1,J+1)=(2.-SIGMA)*F1(I1,J1)/PIG-ARR(I,J)
      GO TO 85100
8520  TMA(1)=1
      LEND=M1-2
      DO 8530 I=1,LEND
8530  TMA(I+1)=TMA(I)+C(2*I-1)*2
      IF(L3.NE.1)GO TO 8540
      FAC1=-1
      FAC2=1
8540  IF(L3.NE.2) GO TO 8550
      FAC1=1
      FAC2=-1
8550  DO 8560 J=1,QMA,2
      J1=(J+1)/2
      IF(J.GE.TMA(M1/2)) GO TO 85200
      I=2
8552  IF(J.LT.TMA(I))GO TO 85301
      I=I+1
      IF(I.LE.M1/2)GO TO 8552
85301 J2=(TMA(M1-I+1)+J-TMA(I-1)+1)/2
      DO 85400 I=1,PMA,2
      I1=(I+1)/2
      ARR(I,J)=((1.-SIGMA)*(F1(I1,J1)+FAC1*F1(I1,J2))
     $+SIGMA*(F3(I1,J1)+FAC1*F3(I1,J2)))/PIG
      ARR(I+1,J)=(SIGMA*(F2(I1,J1)+FAC1*F2(I1,J2)))/PIG
      ARR(I,J+1)=(SIGMA*(F2(I1,J1)+FAC2*F2(I1,J2)))/PIG
      ARR(I+1,J+1)=(F1(I1,J1)+F1(I1,J2)*FAC2-SIGMA*(F3(I1,J1)
     $+F3(I1,J2)*FAC2))/PIG
85400 CONTINUE
      GO TO 8560
85200 DO 8559 I=1,PMA,2
      I1=(I+1)/2
```

```
      ARR(I,J)=((1.-SIGMA)*F1(I1,J1)+SIGMA*F3(I1,J1))/PIG
      ARR(I,J+1)=(SIGMA*F2(I1,J1))/PIG
      ARR(I+1,J)=ARR(I,J+1)
8559  ARR(I+1,J+1)=(2.-SIGMA)*F1(I1,J1)/PIG-ARR(I,J)
8560  CONTINUE
85100 CONTINUE
C
C
      IF(ABS(SX).GT.1.E-4.OR.ABS(SY).GT.1.E-4) GO TO 86111
       GO TO 85100
C     IN-LINE ADS
86111 M2=M1-1
      J=1
      I=1
      NN=0
      SXH=LXN/H
      SYH=LYN/H
      IF(L3.NE.0)M2=M1/2
      DO 87100 I1=1,M2
      L1=2*I1-1
      L=0
      L2=L1+1
      IF(C(L1).GE.C(L2))GO TO 8710
      L=1
      ARR(I,J)=ARR(I,J)+SXH
      ARR(I+1,J+1)=ARR(A+1,J+1)+SYH
      II=I+C(L1)*2
      JJ=J+(C(L1)-1)*2
      ARR(II,JJ)=ARR(II,JJ)-SXH
      ARR(II+1,JJ+1)=ARR(II+1,JJ+1)-SYH
8710  MM=L+NN+1
      NN=NN+C(L2)-L
      IF(MM.GE.NN)GO TO 8730
      DO 8729 I2=MM,NN
      IF(MM.GT.NN)GO TO 8729
      I3=2*I2-1
      I4=I3+1
      ARR(I3,J)=ARR(I3,J)-SXH
      ARR(I3,J+2)=ARR(I3,J+2)+SXH
      ARR(I4,J+1)=ARR(I4,J+1)-SYH
      ARR(I4,J+3)=ARR(I4,J+3)+SYH
8729  J=J+2
8730  CONTINUE
87100 CONTINUE
C
C
86100 KPG=KAPPA/PI/G
      WRITE(3,901)
      IEND=2*M-1
      DO 86110 I=1,IEND,2
      L=(I+1)/2
      RU(I)=UX-PHI*YU(L)+KPG*RZU(I,1)
      RU(I+1)=UY+PHI*XU(L)+KPG*RZU(I+1,1)
86110 CONTINUE
      JEND=2*N-1
      DO 86120 J=1,JEND,2
      L=(J+1)/2
      RT(J)=UX-PHI*YT(L)+KPG*RZT(J,1)
```

```
      RT(J+1)=UY+PHI*XT(L)+KPG*RZT(J+1,1)
86120 CONTINUE
      IF(L3.EQ.0) GO TO 86130
      L=0
      J=0
      IEND=IFIX(FLOAT(M1)/2.0-.9)
      DO 86129 I=1,IEND
      J=J+C(2*I-1)*2
      L=L+C(2*I)
86129 CONTINUE
      DO 86128 I=1,J
86128 RT(I)=RT(I)*2
      DO 86127 I=1,L
86127 ZU(I)=ZU(I)*2
86130 JEND=2*N
      DO 86140 J=1,JEND
86140 T(J,1)=0
      RE=.2
      RB=.2
      MM=0
      B=1.0
      E=.5
C PW0:
86150 JEND=2*N
      DO 86155 J=1,JEND
86155 TT(J)=T(J,1)
      MM=MM+1
      L1=0
C PW1 :
86160 NN=0
C PW2 :
C IN-LINE PENALT
C  DX=1.0, EX=E, PX=P, T=T, MU=MU
C
86170 DXP=1.0
      DO 8799 I=1,N
      GX=MU*MU*ZT(I)*ZT(I)-T(2*I-1,1)*T(2*I-1,1)-T(2*I,1)*T(2*I,1)
      IF(GX)8703,8702,8702
8702  P(2*I-1)=-2*E/(GX+DXP*E)
      P(2*I)=4.*E/((GX+DXP*E)*(GX+DXP*E))
      GO TO 8799
8703  P(2*I-1)=-2./DXP+2.*GX/(DXP*DXP*E)
      P(2*I)=4./(DXP*DXP*E)
8799  CONTINUE
C
C
      JEND=2*N-1
      DO 86175 J=1,JEND,2
      IF (P(J).LT.-1.E10)GO TO 86180
86175 CONTINUE
      GO TO 86190
C PW3:
86180 IF(L1.NE.1)GO TO 86200
      WRITE(3,904)
904   FORMAT(' PROCESS INTERRUPTED, RESULTS MAT NOT BE SIGNIFICANT')
C
      IPCODE=0
C IN-LINE PRINT
```

```
4145  LL=L3
 9901 IF(NS.GT.1) GO TO 499
      WRITE(3,9004)
9004  FORMAT(//,20X,'***** CONTACT REGION FOLLOWS *****',/,
     $10X,'X AND Y ARE NORMALIXED COORDINATES, X IN THE ROLLING',/,
     $10X,'DIRECTION, X,Y=X1/C1,Y1/C1 WHERE X1,Y1 ARE DIM. COORD.',/,
     $10X,'TZH=HERTZ STRESS =3/(2*PI)*SQRT(1.0-X*X/(A*A)-Y*Y/(B*B))'
     $,/,10X,'TX AND TY ARE NORMALIZED SHEAR STRESSES',/,10X,'TX=-TAUXZ*
     $C**3/(RHO*N), TY=-TAUYZ*C**3/(RHO*N)',/,
     $10X,'ABS(TX,TY) LESS THAN TZH FOR NO SLIP, EQUAL TO TZH FOR SLIP',
     $/,10X,'VX,VY ARE NORMALIZED SLIP COMPONENTS, VX=VX1/V*RHO/(MU*C)',
     $/,10X,'VY=VY1/V*RHO/(MU*C), WHERE VX1,VX2=REL. VEL. BETWEEN',/,10X
     $,'ADJACENT POINTS AND V=ROLLING VEL.',////)
  499 CONTINUE
      LU=1
      LT=1
      J1=1
      FAC1P=1
      FAC2P=1
      WP=0
      J=M1/2
      IF(LL.EQ.0)J=M1-1
C     V2:
8802  DO 8801 I=J1,J
      IF(J1.GT.J)GO TO 8801
      MAX=C(2*I-1)
      L3P=2
      IF(C(2*I-1).GE.C(2*I))GO TO 8803
      MAX=C(2*I)
      L3P=1
 8803 CONTINUE
909   FORMAT(/)
      IF(WP.NE.1)GO TO 8804
      LU=LU-C(2*I-2)-C(2*I)
      LT=LT-C(2*I-3)-C(2*I-1)
8804  FIX1=YT(LT)*FAC1P*FAC2P
      IBLANK=0
      DO 8801 I1=1,MAX
      IF(L3P.EQ.2)GO TO 8812
C     SS1:
8800  TX=U(2*LU-1,1)*FAC1P
      TY=U(2*LU,1)*FAC2P
      FIX3=SQRT(TX*TX+TY*TY)
      IF(ABS(U(2*LU-1,1)).LT.1.E-20)TX=1.E-20
      FIX2=180./PI*ATAN(TY/TX)+(1.0-SIGN(TX))*90.
      IF(NS.GT.1) GO TO 501
      IF(IBLANK.EQ.0)WRITE(3,9006)FIX1
      IF(IBLANK.EQ.0) WRITE(3,9009)
      WRITE(3,9008) XU(LU),FIX3,FIX2
 9008 FORMAT(1X,1F11.4,33X,2F11.4)
  501 CONTINUE
      IBLANK=1
      LU=LU+1
      IF(MAX.EQ.C(2*I).AND.I1.EQ.MAX)GO TO 8813
C     SS2:
8812  FIX1A=XT(LT)
      TX=T(2*LT-1,1)*FAC1P
      TY=T(2*LT,1)*FAC2P
```

```
      FIX2=TX
      FIX3=TY
      FIX4=SQRT(TX*TX+TY*TY)
      IF(ABS(T(2*LT-1,1)).LT.1.E-20)TX=1.E-20
      FIX5=180./PI*ATAN(TY/TX)+(1.-SIGN(TX))*90.
      FIX6=MU*ZT(LT)
      IF(NS.GT.1) GO TO 502
      IF(IBLANK.EQ.0)WRITE(3,9006)FIX1
      IF(IBLANK.EQ.0) WRITE(3,9009)
      WRITE(3,9011) FIX1A,FIX6,FIX4,FIX5
 9011 FORMAT(1X,4F11.4)
  502 CONTINUE
      IBLANK=1
      LT=LT+1
      L3P=1
C     SS3:
8813  CONTINUE
8801  CONTINUE
      IF(LL.EQ.1.AND.WP.EQ.0)GO TO 8859
      GO TO 8850
8859  FAC1P=-1
      WP=1
      J1=M1/2+1
      J=M1-1
      GO TO 8802
8850  IF(LL.EQ.2.AND.WP.EQ.0)GO TO 5188
      GO TO 8851
5188  FAC2P=-1
      WP=1
      J1=M1/2+1
      J=M1-1
      GO TO 8802
8851  MZ=0
      TX=0
      TY=0
      IF(LL.NE.0)GO TO 8852
      JLAST=2*N-1
      DO 8853 J=1,JLAST,2
      TX=TX+T(J,1)
      TY=TY+T(J+1,1)
      MZ=MZ+XT((J+1)/2)*T(J+1,1)-YT((J+1)/2)*T(J,1)
8853  CONTINUE
      GO TO 8855
8852  LT=1
      ILAST=FLOAT(M1)/2.0-0.9
      DO 8856 I=1,ILAST
      JLAST=2*C(2*I-1)
      DO 8856 J=1,JLAST,2
      TX=TX+(1+FAC1P)*T(LT,1)
      TY=TY+(1+FAC2P)*T(LT+1,1)
      MZ=MZ+XT((LT+1)/2)*T(LT+1,1)*(FAC2P+1)-YT((LT+1)/2)*
     $T(LT,1)*(-FAC1P+1)
      LT=LT+2
8856  CONTINUE
      JLAST=2*C(M1-1)
      DO 8858 J=1,JLAST,2
      TX=TX+T(LT,1)
      TY=TY+T(LT+1,1)
```

```
      MZ=MZ+XT((LT+1)/2)*T(LT+1,1)-YT((LT+1)/2)*T(LT,1)
      LT=LT+2
8858  CONTINUE
C     ZOZO:
8855  TX=TX*H*K
      TY=TY*H*K
      MZ=MZ*H*K
      RES=SQRT(TX**2+TY**2)
      WRITE(3,905)
905   FORMAT(///)
      WRITE(3,977)TX,TY,RES
      WRITE(3,978)MZ
      IF(IPCODE.EQ.1)GO TO 6470
      GO TO 9999
86200 L1=1
      JLAST=2*N
      DO 7110 J=1,JLAST
7110  T(J,1)=TT(J)
      RB=SQRT(RB)
      RE=SQRT(RE)
      B=B/RB
      E=E/RE
      GO TO 86160
C PW4: IN-LINE NEWTON
86190 MM=2*M-1
      NNN=2*N-1
      N2=NNN+1
      EPS=1.E-15
C
      CALL ARRAY(2,2*M,2*N,120,120,ARR,ARR)
      CALL ARRAY(2,2*N,1,120,1,T,T)
      CALL GMPRD(ARR,T,U,2*M,2*N,1)
      CALL ARRAY(1,2*M,2*N,120,120,ARR,ARR)
      CALL ARRAY(1,2*N,1,120,1,T,T)
      CALL ARRAY(1,2*M,1,120,1,U,U)
C     ABOVE IS EQUIVALENT TO CALL TO MATVER(A,T,U)
      DO 6910 I=1,MM,2
      II=(I+1)/2
      U(I,1)=U(I,1)+RU(I)
      U(I+1,1)=U(I+1,1)+RU(I+1)
      S(II)=SQRT(U(I+1,1)*U(I+1,1)+U(I,1)*U(I,1)+B)
      U1(1,I)=(MU*ZU(II)*U(I,1))/S(II)
      U1(1,I+1)=(MU*ZU(II)*U(I+1,1))/S(II)
6910  CONTINUE
C
      CALL ARRAY(2,1,2*M,1,120,U1,U1)
      CALL ARRAY(2,2*M,2*N,120,120,ARR,ARR)
      CALL GMPRD(U1,ARR,FACC,1,2*M,2*N)
      CALL ARRAY(1,1,2*M,1,120,U1,U1)
      CALL ARRAY(1,2*M,2*N,120,120,ARR,ARR)
      CALL ARRAY(1,1,2*N,1,120,FACC,FACC)
C      ABOVE IS EQUIVALENT TO MATVER(U1,A,FACC)
      DO 6920 J=1,NNN,2
      FACC(1,J)=-FACC(1,J)+RT(J)+P(J)*T(J,1)
      FACC(1,J+1)=-FACC(1,J+1)+RT(J+1)+P(J)*T(J+1,1)
6920  CONTINUE
      DO 6940 I=1,M
      I2=2*I
```

```
      I1=I2-1
      SS=S(I)*S(I)
      MZS=(MU*ZU(I))/S(I)
      K1=MZS*(1-U(I1,1)*U(I1,1)/SS)
      K2=-MZS*J(I1,1)*U(I2,1)/SS
      K3=MZS*(1-U(I2,1)*U(I2,1)/SS)
      DO 6930 J=1,N2
      U2(J,I1)=K1*ARR(I1,J)+K2*ARR(I2,J)
      U2(J,I2)=K2*ARR(I1,J)+K3*ARR(I2,J)
6930  CONTINUE
6940  CONTINUE
      CALL ARRAY(2,2*N,2*M,120,120,U2,U2)
      CALL ARRAY(2,2*M,2*N,120,120,ARR,ARR)
      CALL GMPRD(U2,ARR,FDACC,2*N,2*M,2*N)
      CALL ARRAY(1,2*N,2*M,120,120,U2,U2)
      CALL ARRAY(1,2*M,2*N,120,120,ARR,ARR)
      CALL ARRAY(1,2*N,2*N,120,120,FDACC,FDACC)
C      ABOVE IS EQUIVALENT TO MATVER(U2,A,FDACC)
      DO 6950 I=1,NNN,2
      FDACC(I,I)=FDACC(I,I)-P(I)+P(I+1)*T(I,1)*T(I,1)
      TEMP=FDACC(I,I+1)+P(I+1)*T(I,1)*T(I+1,1)
      FDACC(I,I+1)=TEMP
      FDACC(I+1,I)=TEMP
      FDACC(I+1,I+1)=FDACC(I+1,I+1)-P(I)+P(I+1)*T(I+1,1)*T(I+1,1)
6950  CONTINUE
      IF (L3.EQ.0) GO TO 6960
      J=1
      ILAST=FLOAT(M1)/2.0-0.9
      DO 6980 I=1,ILAST
6980  J=J+C(2*I-1)*2
      I2=1
      IF(L3.EQ.1)I2=0
      IFIRST=J+I2
      ILAST=J+C(M1-1)*2-1
      DO 6970 I=IFIRST,ILAST,2
      IF(IFIRST.GT.ILAST)GO TO 6970
      FACC(1,I)=0
      DO 6970 I1=1,N2
      FDACC(I,I1)=0
      FDACC(I1,I)=0
      FDACC(I,I)=1
6970  CONTINUE
C PAS:
6960  CONTINUE
      CALL ARRAY(2,1,2*N,1,120,FACC,FACC)
      CALL ARRAY(2,2*N,2*N,120,120,FDACC,FDACC)
      CALL GELG(FACC,FDACC,N2,1,EPS,IER)
      CALL ARRAY(1,1,2*N,1,120,FACC,FACC)
      CALL ARRAY(1,2*N,2*N,120,120,FDACC,FDACC)
C     ABOVE IS EQUIVALENT TO ADGELG(FACC,FDACC,N2,1,EPS,IER)
C
      IF(IER) 6989,6990,6989
6989  WRITE(3,6901)
6901  FORMAT(//' SINGULAR MATRIX, NO RESULTS'//)
      GO TO 9999
6990  DO 6999 J=1,N2
      T(J,1)=T(J,1)+FACC(1,J)
      L=1
```

```
      DO 6999 I=1,N2
      IF(ABS(FACC(1,I)).GE.1.E-4) L=0
6999  CONTINUE
C     END OF NEWTON
C
      NN=NN+1
      IF(NN.LT.20) GO TO 86191
      GO TO 86180
86191 IF(L.EQ.0) GO TO 86170
      IF(B.LT.1.E-8.AND.E.LT.1.E-8)GO TO 29168
      GO TO 86192
29168 IPCODE=1
      GO TO 4145
6470  CONTINUE
      GO TO 9999
86192 IF(B.GT.1.E-8)B=B*RB
      IF(E.GT.1.E-8)E=E*RE
      GO TO 86150
C VOLG:
9999  L6=L3
      IF(L3.EQ.0)GO TO 9991
      L=0
      J=0
      ILAST=FLOAT(M1)/2.-.9
      DO 9990 I=1,ILAST
      J=J+C(2*I-1)*2
      L=L+C(2*I)
9990  CONTINUE
      DO 9992 I=1,J
9992  RT(I)=RT(I)*.5
      DO 9993 I=1,L
9993  ZU(I)=ZU(I)*.5
9991  CONTINUE
      N=L4
      M=L5
  997 CONTINUE
      GO TO 999
  998 WRITE(3,979)
999   CONTINUE
 9006 FORMAT(/,3X,'*** Y=',1F11.4)
9009  FORMAT( 7X,'X',10X,'TZH',        5X,'ABS(TX,TY)',1X,'ARG(TX,TY)',
     $       1X,'ABS(VX,VY)',1X,'ARG(VX,VY)')
 968  FORMAT('1',///,T63,'PROGRAM WISK-SRT',/,T54,'GENERAL THEORY OF
     $ROLLING CONTACT',/,T64,'BY J.J. KALKER',/,T56,'MODIFIED AT CLEMSON
     $ UNIVERSITY',/,T61,'DEPT. OF MECH. ENGR.',/,T66,'CLEMSON, SC',//)
 969  FORMAT(////,58X,'***** INPUT PARAMETERS *****',//)
 970  FORMAT(16X,'NORMALIZED CONTACT DIMENSIONS        A=',1PE11.4,10X,'(
     $ A=A1/C1, B=B1/C1, WHERE C1=SQRT(A1*B1),',/,32X,'(CARD #2)'
     $,11X,'B=',1PE11.4,10X,'( A1,B1 ARE ACTUAL CONTACT DIMENSIONS',//,
     $19X,'  COMBINED POISSON S RATIO       NU=',1PE11.4,/,33X,'(CARD #2)
     $',//,28X,'LAYER STIFFNESSES      LXN=',1PE11.4,/,33X,'(CARD #2)',
     $ 8X,'LYN=',1PE11.4,/,21X,'       ELASTIC DIFFERENCE   KAPPA=',1PE11
     $.4,/,33X,'(CARD #2)',/)
 972  FORMAT( 26X,'NUMERICAL CONSTANTS       N1=',I3,/,31X,'(CARD #3)',
     $11X,'M1=',I3,/,51X,'NS=',I3,//)
 973  FORMAT(47X,'***** PARAMETERS COMPUTED AND USED IN PROGRAM *****'
     $,//,    21X,'NORMALIZED SHEAR MODULUS        GS=',1PE11.4,/,22X,'(CO
     $MBINED)',//,52X,'N=',I3,5X,'N=NUMBER OF TRACTION POINTS',/,
```

```
     $52X,'M=',I3,5X,'M=NUMBER OF SLIP POINTS',//)
 974  FORMAT(42X,'***** NV2=',I2,' DISTINCT PROBLEMS FOLLOW FOR DIFFEREN
     $T *****',/,45X,'***** VALUES OF NORMALIZED CREEPAGE AND SPIN *****
     $',//)
 975  FORMAT(//,17X,'NORMALIZED CREEPAGE AND SPIN     UXN=',1PE11.4,/,
     $23X,'(INPUT ON CARD #5)',
     $ 9X,'UYN=',1PE11.4,/,50X,'PHN=',1PE11.4,//)
 977  FORMAT( 24X,'NORMALIZED FORCES ARE     FXN=',1PE11.4,/,29X,
     $'(COMPUTED)',11X,'FYN=',1PE11.4,//,24X,'RESULTANT FORCE
     $RES=',1PE11.4,/,24X,'(RES=SQRT(FXN**2+FYN**2))',//)
 978  FORMAT( 25X,'NORMALIZED MOMENT IS     MZN=',1PE11.4,/,
     $30X,'(COMPUTED)',//)
      STOP
  979  FORMAT(//,58X,'***** A/B LESS THAN 0.1 *****',/,
     $58X,'***** WORK NEXT PROBLEM *****',//)
      END
```

SIGN

```
      FUNCTION SIGN(X)
      IF(X)10,20,30
10    SIGN=-1.0
      RETURN
20    SIGN=0
      RETURN
30    SIGN=1.0
      RETURN
      END
```

CONST

```
      SUBROUTINE CONST(A,B,NU,GS)
      DIMENSION D(3),E(3,20),AR(20)
C     ***** DATA E(I,J) GIVES THE VALUES OF GS FROM
C     ***** KALKER'S TABLE, VALID FOR A/B EQUAL TO OR GREATER THAN 0.1
      REAL NU
      DATA E/
     $ 0.7670, 0.5752, 0.3835, 0.5608, 0.4206, 0.2804, 0.4779, 0.3584,
     $ 0.2390, 0.4343, 0.3257, 0.2172, 0.4089, 0.3066, 0.2044, 0.3934,
     $ 0.2950, 0.1967, 0.3840, 0.2880, 0.1920, 0.3785, 0.2839, 0.1892,
     $ 0.3758, 0.2818, 0.1879, 0.3750, 0.2812, 0.1875, 0.3758, 0.2818,
     $ 0.1879, 0.3785, 0.2839, 0.1892, 0.3840, 0.2880, 0.1920, 0.3934,
     $ 0.2950, 0.1967, 0.4089, 0.3066, 0.2044, 0.4343, 0.3257, 0.2172,
     $ 0.4779, 0.3584, 0.2390, 0.5608, 0.4206, 0.2804, 0.7670, 0.5752,
     $ 0.3835, 0.7918, 0.5938, 0.3959/
      DATA AR / 0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0,1.111111,
     $1.25,1.428571,1.666667,2.0,2.5,3.333333,5.0,10.0,11.0/
      PI=3.14159
      RG=A/B
      IF(RG.GT.AR(20)) GO TO 14
      GO TO 15
   14 SG=B/A
      GS=3.0*(1.0-NU)/(4.0*PI*SQRT(SG))
      GO TO 80
   15 DO 20 I=2,20
      IF(RG.LE.AR(I)) GO TO 25
   20 CONTINUE
   25 J=I
      DO 30 I=1,3
   30 D(I)=E(I,J-1)+(E(I,J)-E(I,J-1))*(RG-AR(J-1))/(AR(J)-AR(J-1))
      AL=8.0*(D(3)-2.0*D(2)+D(1))
      BE=2.0*(-D(3)+4.0*D(2)-3.0*D(1))
      GS=AL*NU**2+3E*NU+D(1)
   80 CONTINUE
      RETURN
      END
```

```
                      PROGRAM WISK-SRT
          GENERAL THEORY OF    ROLLING CONTACT
                      BY J.J. KALKER
             MODIFIED AT CLEMSON UNIVERSITY
                   DEPT. OF MECH. ENGR.
                       CLEMSON, SC


                ***** INPUT PARAMETERS *****


NORMALIZED CONTACT DIMENSIONS      A= 2.5980E+00          ( A=A1/C1, B=B1/C1, WHERE C1=SQRT(A1*
          (CARD #2)                B= 3.8490E-01          ( A1,B1 ARE ACTUAL CONTACT DIMENSIONS

   COMBINED POISSON S RATIO       NU= 2.8000E-01
           (CARD #2)

         LAYER STIFFNESSES       LXN= 0.0
           (CARD #2)             LYN= 0.0
        ELASTIC DIFFERENCE     KAPPA= 0.0
           (CARD #2)

      NUMERICAL CONSTANTS         N1=  6
          (CARD #3)               M1=  6
                                  NS=  1


         ***** PARAMETERS COMPUTED AND USED IN PROGRAM *****

   NORMALIZED SHEAR MODULUS       GS= 4.5572E-01
    (COMBINED)

                                   N= 25     N=NUMBER OF TRACTION POINTS
                                   M= 26     M=NUMBER OF SLIP POINTS


       ***** NV2= 1 DISTINCT PROBLEMS FOLLOW FOR DIFFERENT *****
         ***** VALUES OF NORMALIZED CREEPAGE AND SPIN *****


 NORMALIZED CREEPAGE AND SPIN    UXN= 0.0
      (INPUT ON CARD #5)         UYN=-1.4000E+00
                                 PHN= 8.0000E-01
```

PROCESS INTERRUPTED, RESULTS MAT NOT BE SIGNIFICANT

***** CONTACT REGION FOLLOWS *****

X AND Y ARE NORMALIXED COORDINATES, X IN THE ROLLING
DIRECTION, X,Y=X1/C1,Y1/C1 WHERE X1,Y1 ARE DIM. COORD.
TZH=HERTZ STRESS =3/(2*PI)*SQRT(1.0-X*X/(A*A)-Y*Y/(B*B))
TX AND TY ARE NORMALIZED SHEAR STRESSES
TX=-TAUXZ*C**3/(RHO*N), TY=-TAUYZ*C**3/(RHO*N)
ABS(TX,TY) LESS THAN TZH FOR NO SLIP, EQUAL TO TZH FOR SLIP
VX,VY ARE NORMALIZED SLIP COMPONENTS, VX=VX1/V*RHO/(MU*C)
VY=VY1/V*RHO/(MU*C), WHERE VX1,VX2=REL. VEL. BETWEEN
ADJACENT POINTS AND V=ROLLING VEL.

*** Y= 0.2566

X	TZH	ABS(TX,TY)	ARG(TX,TY)	ABS(VX,VY)	ARG(VX,VY)
-1.7320	0.1592	0.1592	265.6077		
-1.2990				2.6082	264.6062
-0.8660	0.3183	0.4744	233.2914		
-0.4330				1.7799	261.4832
0.0	0.3559	0.3559	253.3449		
0.4330				0.5797	234.1627
0.8660	0.3183	0.3183	215.1041		
1.2990				0.0606	172.6306
1.7320	0.1592	0.1592	117.2705		

*** Y= 0.1283

X	TZH	ABS(TX,TY)	ARG(TX,TY)	ABS(VX,VY)	ARG(VX,VY)
-2.1650				3.5929	268.2097
-1.7320	0.3183	0.3184	266.5671		
-1.2990				2.5799	264.9429
-0.8660	0.4211	0.4211	219.3307		
-0.4330				1.8049	269.0361
0.0	0.4502	0.4645	251.9182		
0.4330				0.3455	226.5659
0.8660	0.4211	0.3143	147.3318		
1.2990				0.0001	98.3786
1.7320	0.3183	0.3183	117.6059		
2.1650				0.0281	123.1223

*** Y= 0.0000

X	TZH	ABS(TX,TY)	ARG(TX,TY)	ABS(VX,VY)	ARG(VX,VY)
-2.1650				3.6267	269.9998
-1.7320	0.3559	0.3559	-90.0000		
-1.2990				2.5857	270.0000
-0.8660	0.4502	0.4502	-90.0000		
-0.4330				1.7915	269.9998
0.0	0.4775	0.4775	-90.0000		
0.4330				0.1718	-89.9990
0.8660	0.4502	0.2224	90.0000		
1.2990				0.0000	260.3901
1.7320	0.3559	0.3559	90.0000		
2.1650				0.0156	269.9873

*** Y= -0.1283

X	TZH	ABS(TX,TY)	ARG(TX,TY)	ABS(VX,VY)	ARG(VX,VY)
-2.1650				3.5929	-88.2097
-1.7320	0.3183	0.3184	-86.5672		
-1.2990				2.5799	-84.9430
-0.8660	0.4211	0.4211	-39.3307		
-0.4330				1.8049	-89.0363
0.0	0.4502	0.4645	-71.9182		
0.4330				0.3455	-46.5659
0.8660	0.4211	0.3143	32.6682		
1.2990				0.0001	81.6214
1.7320	0.3183	0.3183	62.3941		
2.1650				0.0281	56.8777

*** Y= -0.2566

X	TZH	ABS(TX,TY)	ARG(TX,TY)	ABS(VX,VY)	ARG(VX,VY)
-1.7320	0.1592	0.1592	-85.6078		
-1.2990				2.6082	-84.6064
-0.8660	0.3183	0.4744	-53.2914		
-0.4330				1.7799	-81.4834
0.0	0.3559	0.3559	-73.3449		
0.4330				0.5797	-54.1627
0.8660	0.3183	0.3183	-35.1041		
1.2990				0.0606	7.3694
1.7320	0.1592	0.1592	62.7295		

NORMALIZED FORCES ARE (COMPUTED) FXN= 0.0
FYN=-4.1081E-01

RESULTANT FORCE RES= 4.1081E-01
(RES=SQRT(FXN**2+FYN**2))

NORMALIZED MOMENT IS (COMPUTED) MZN= 7.3743E-01

APPENDIX B

Reprint of Reference [1]

J. J. Kalker, "Simplified Theory of Rolling Contact", Delft Progress Report, Series C: Mechanical and Aeronautical Engineering and Shipbuilding, 1 (1973). pp. 1-10. Thanks are due the Delft University Press and Professor Kalker for granting permission to include this paper in the report.

ERRATA

Simplified Theory of Rolling Contact [1]

The following errors have been noted in Kalker's paper [1].

1. The right hand side of equations (a) and (b) on page 4 should read ν as shown.
2. The right hand side of equation (17) on page 5 should read $-2fZ_0x/a^2$ as shown.
3. The left hand side of equation (20) on page 6 should read $v_x\{L(y)-x\}/S_x$ as shown.
4. The coefficient of friction is denoted by f and μ interchangeably.
5. Equation (30) on page 7 should read $S_x(v_y + \phi x)\sin(\theta) +$ as shown.
6. Equation (44) on page 9 should read $S_y = 8a/(3C_{22}G)$,
   $h = \frac{32}{3\pi} \cdot \sqrt{\left(\frac{b}{a}\right)} \cdot \frac{C_{23}}{C_{22}}$.
7. Equation (47) on page 10 should read
   $h_o = h(44) = \frac{32}{3\pi} \cdot \sqrt{\left(\frac{b}{a}\right)} \cdot \frac{C_{23}}{C_{22}}$.

Simplified theory of rolling contact

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Delft Progr. Rep., Series C: Mechanical and aeronautical engineering and shipbuilding, **1** (1973) pp. 1–10.

In the present paper an approximate theory of rolling contact of elastic bodies is developed which is very simple to use. All salient features of rolling contact phenomena, with the exception of the phenomena due to elastic asymmetry, are well reproduced. As a consequence it is not difficult to give the parameters of the simplified theory such values that a reasonable quantitative agreement with the exact theory of steady-state rolling is obtained. Finally the simplified theory is well suited to roughly investigate the mechanical influence of the surface layers which may cover the bodies.

Introduction

In the present paper two dry bodies are considered which roll over each other. In first instance the bodies may be regarded as rigid. Then, according to Coulomb's law of dry friction, two states are possible, viz.

1. The bodies roll without slip, and the tangential force falls below a fixed multiple of the normal force by which the bodies are pressed together.
2. The bodies slide and roll while the tangential force attains the fixed multiple of the normal force and acts in the direction of the slip.

However, it has been observed experimentally, that the bodies slip a little even when the force transmitted is below the maximum. In some applications, such as the investigation into the stability of railway trains, these effects are significant and the crude model described above cannot be used.

For an explanation, the elasticity of the contacting bodies must be taken into account. This has been done by several authors, – refer to the bibliography at the end of this paper –, and the theory becomes quite formidable, owing to the complexity of the relationships even of classical elasticity.

In this paper the model is simplified in the sense that these complicated relations are replaced by a much simpler relationship, which appears to conserve many of the typical features of the conventional contact theory. Thus it has illustrative value. Also it appears to be possible to utilise the simplified theory as an approximation of the more realistic, complicated model by adapting certain constants. A program implementing the simplified theory does a job in approximately 1/100 of the time needed for the same job by a program implementing the realistic, complicated model. Thus the simplified theory has a great practical value also.

Formulation of the problem

Consider two elastic bodies which are pressed together so that a contact area forms between them, see Fig. 1. A cartesian coordinate system $\{0; x, y, z\}$ is introduced of which the plane of x and y is the plane of contact and in which the z-axis points vertically downward into

Notations

The exact model: the realistic complicated model.

$$' : \frac{\partial}{\partial x} \cdot \quad \dot{} : \frac{\partial}{\partial t}$$

1, 2: if a distinction must be made between quantities of body 1 or 2, the quantities in question carry a superscript 1 or 2.

(x, y, z): Cartesian coordinate system with origin in centre of the contact area, x- direction coincides with rolling-direction, z points vertically down ward into 2. (see Fig. 1)

$A(y)$, $B(y)$	(14. III)	$\mathbf{u}, \mathbf{u}^1, \mathbf{u}^2$	above (3)
a, b	(1)	u_x, u_y, u_z	(7a)
C	(1)	V	(5)
C_{ij}	(13)	$\mathbf{V}, \mathbf{V}^1, \mathbf{V}^2$	(3)
F_x, F_y	(13a, b)	$\mathbf{V}_r^1, \mathbf{V}_r^2$	above (3) (5), (6)
f	(2)	$\mathbf{v}$	(2), (10)
G	(13)	X, Y	(2)
H	(30)	Z	(1)
h	(44), (47)	Z_0	(14. I, II, III)
$L(y)$	(12)	δ	(38)
l	(35)	θ	(25a)
M_z	(13c)	θ_0	(32)
N	(40)	λ	(25b)
S_x, S_y	(9) and below	$\upsilon(t)$, $\upsilon(l)$	$=\upsilon_x$
$\mathbf{s}(s_x, s_y)$	(7b)	$\upsilon_x, \upsilon_y, \phi$	(6)
t	time	$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yy}, \tau_{yx}$	stresses

body 2, see Fig. 1. The origin is the centre of the contact area.

We assume that the contact area C and the normal pressure Z acting on it can be calculated by means of the Hertz theory. For this it is sufficient that:

1. the small displacement, small displacement gradient theory of elasticity is applicable;
2. the largest diameter of the contact area is small with respect to a characteristic linear dimension of the bodies at and near the contact area;
3. no close conformity may exist between the bodies at the contact area;
4. the bodies must be homogeneous in the parts that are sensibly affected by the elastic deformation;
5. either: a. the bodies are made of identical materials, or they are incompressible

or: b. the level of the surface shear tractions (X, Y) is at each point of the contact area much lower than that of the normal pressure: $\|X, Y\| \ll Z$. For this it is sufficient that the coefficient of friction $f \ll 1$.

According to the Hertz theory (see ref. 1 p. 193 sqq) the contact area C is elliptical in shape, and the pressure acting over it is ellipsoidal:

$$C = \{x,y,z : z = 0, x^2/a^2 + y^2/b^2 \leq 1\} \quad \text{contact area}$$

$$Z(x,y) = -\sigma_z = 0 \text{ on } z = 0, \text{ outside } C; \quad = Z_0\sqrt{(1 - x^2/a^2 - y^2/b^2)} \text{ inside } C. \tag{1}$$

The Hertz theory does not consider the surface shear traction (X, Y), but the surface shear traction, which will be called tangential traction, is an important object of study in this paper. The tangential traction (= force/unit area) (X, Y) exerted by body 1 or body 2 vanishes on the surface of the bodies outside C, and inside C it is governed by the Coulomb law of dry friction which connects the slip $\mathbf{v}$ of body 1 over body 2 with the tangential traction (X, Y). First it is observed that there is no vertical (z) component of the slip, since no gap forms at a point remaining in the contact area. So the z-component of the velocities which occur are left out of consideration. Coulomb's law of friction reads:

$\mathbf{v}$ = velocity of body 1 over body 2 =
$= 0 \rightarrow \|(X, Y)\| \leq fZ$, adhesion area.
f: coeff. of friction, taken constant
$\mathbf{v} \neq 0 \rightarrow (X, Y) = fZ\mathbf{v}/\|\mathbf{v}\|$, slip area
(X, Y) = tangential traction exerted by body 1 on body 2 = $(-\tau_{xz}, -\tau_{yz})$. (2)

It is seen that the slip is of prime importance in the boundary conditions, and we proceed to find an expression for it. A particle that lies in (x, y, z) in the unstressed state lies in $(x+u_x, y+u_y, z+u_z)$ in the deformed state, where we denote by $\mathbf{u}(u_x, u_y, u_z)$ the elastic displacement of the particle. We find the velocity $\mathbf{V}$ of the particle in the deformed state by differentiating the position with respect to the time t. If we write $\mathbf{V}_r$ for the velocity of the particle in the undeformed state, we obtain the following Eulerian equation

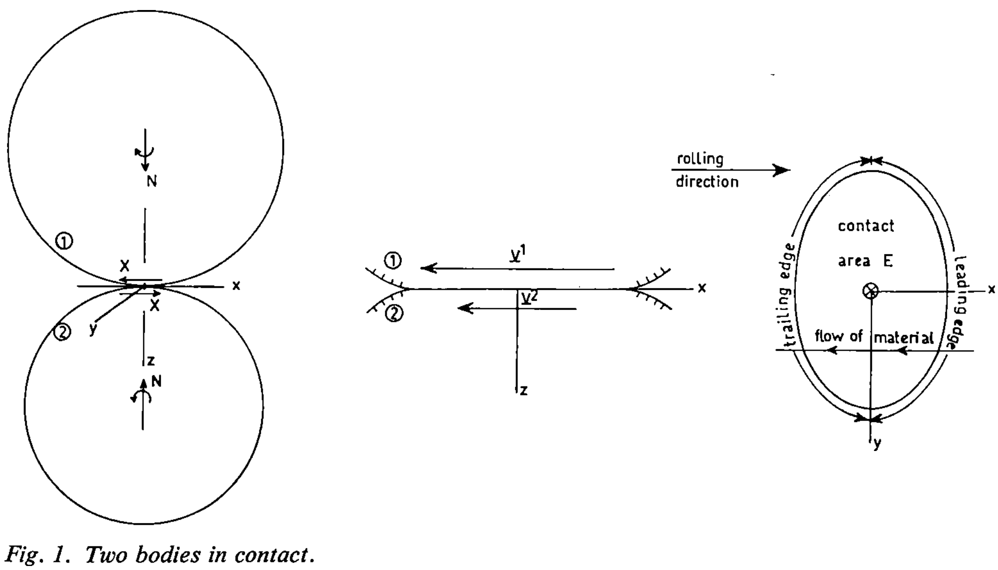

Fig. 1. Two bodies in contact.

$$\mathbf{V} = \mathbf{V}_r + \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{V}_r \cdot \text{grad})\,\mathbf{u} \qquad (3)$$

The slip of body 1 with respect to body 2 is gives by

$$\mathbf{V}^1 - \mathbf{V}^2 = (\mathbf{V}_r^1 - \mathbf{V}_r^2) + \frac{\partial(\mathbf{u}^1 - \mathbf{u}^2)}{\partial t} +$$

$$+ \tfrac{1}{2}\{(\mathbf{V}_r^1 - \mathbf{V}_r^2)\cdot \text{grad}\}\,(\mathbf{u}^1 + \mathbf{u}^2) +$$

$$+ \tfrac{1}{2}\{(\mathbf{V}_r^1 + \mathbf{V}_r^2)\cdot \text{grad}\}\,(\mathbf{u}^1 - \mathbf{u}^2).$$

In this equation we may neglect the third term of the right-hand side compared with the first term, since the displacement gradients were assumed to be small with respect to unity. This gives

$$\mathbf{V}^1 - \mathbf{V}^2 = (\mathbf{V}_r^1 - \mathbf{V}_r^2) + \frac{\partial(\mathbf{u}^1 - \mathbf{u}^2)}{\partial t} +$$

$$+ \tfrac{1}{2}\{(\mathbf{V}_r^1 + \mathbf{V}_r^2)\cdot \text{grad}\}\,(\mathbf{u}^1 - \mathbf{u}^2) \qquad (4)$$

In the steady rolling of two bodies of revolution, rolling takes place approximately in the direction of the parallel circles, that is, almost in the direction of one of the axes of the contact ellipse. In practise, the vast majority of cases to be investigated is of this type, so that we do not lose much if we confine ourselves to the case that the rolling direction nearly coincides with one of the axes of the contact ellipse C, say the positive x-axis. We take our coordinate system in such a way that the origin remains at the centre of the contact ellipse. The material of the bodies near the contact area then flows through the coordinate system almost in the direction of the negative x-axis, with a velocity equal to the rolling speed, see Fig. 1.
So we can identify $\frac{1}{2}(\mathbf{V}_r^1 + \mathbf{V}_r^2)$ with the opposite of the rolling velocity. Since in Eq. (4) for the slip the vector $\frac{1}{2}(\mathbf{V}_r^1 + \mathbf{V}_r^2)$ is multiplied with the small quantity grad $(\mathbf{u}^1 - \mathbf{u}^2)$, we only need the principal term of $\frac{1}{2}(\mathbf{V}_r^1 + \mathbf{V}_r^2)$,

$$\tfrac{1}{2}(\mathbf{V}_r^1 + \mathbf{V}_r^2) \simeq (-\,\mathrm{V}, 0) \qquad (5)$$

where V is the rolling velocity which is greater than zero.
The difference of the velocities of the undeformed surfaces can be regarded as a translation and a rotation, thus

$$\mathbf{V}_r^1 - \mathbf{V}_r^2 = V(\upsilon_x - \phi y,\ \upsilon_y + \phi x) \qquad (6)$$

We call υ_x the longitudinal creepage, υ_y the lateral creepage, and ϕ the spin. The terms creep and creep ratio are also used in the literature for the creepage. Introduction of (5) and (6) into (4) gives for the slip

$\mathbf{u} = \mathbf{u}^2 - \mathbf{u}^1 = (u_x, u_y, u_z)$; displacement difference

$$\mathbf{v} = \mathbf{V}^1 - \mathbf{V}^2 = \Big(V\upsilon_x - V\phi_y - \frac{\partial u_x}{\partial t} + V\frac{\partial u_x}{\partial x},\ V\upsilon_y + V\phi_x - \frac{\partial u_y}{\partial t} + V\frac{\partial u_y}{\partial x} \Big) \qquad (7a)$$

In the slip, the z-component has been left out, since a non-zero vertical (z) component would mean either that contact is broken, or that the bodies penetrate.
A quantity frequently used instead of the slip is the relative slip s

$$\mathbf{s}(s_x, s_y) = \mathbf{v}/V = \Big(\upsilon_x - \phi y - \frac{1}{V}\frac{\partial u_x}{\partial t} + \frac{\partial u_x}{\partial x},\ \upsilon_y + \phi x - \frac{1}{V}\frac{\partial u_y}{\partial t} + \frac{\partial u_y}{\partial x} \Big) \qquad (7b)$$

In steady rolling, the displacement $\mathbf{u}$ is independent of the time, so that the relative slip becomes

$$\mathbf{s}(s_x, s_y) = \Big(\upsilon_x - \phi y + \frac{\partial u_x}{\partial x},\ \upsilon_y + \phi x + \frac{\partial u_y}{\partial x} \Big) \qquad (8)$$

(steady rolling)

which is independent of the rolling velocity.
A complicated relationship (see ref. 2 p. 17sqq, Ref. 1 p. 243) connects the displacement $\mathbf{u}$ with the traction (X, Y) exerted by body (1) on body (2). This relationship will be simplified by putting

$$u_x = S_x X,\ u_y = S_y Y;$$
$$X = -\tau_{xz},\quad Y = -\tau_{yz} \text{ at } z = 0 \qquad (9)$$

where S_x and S_y are the weaknesses in the x and y directions. *The simplification of the simplified theory with respect to the exact theory consists of the adoption of (9) as the traction-displacement relation instead of the* *exact relation described in Ref. 1 p.243 and Ref. 2 p. 17 sqq.* (9) is the response to shear traction of a very thin elastic layer, mounted on a rigid substrate.

It should be noted that Eq. (9) is only an approximation of the true state of affairs if the bodies are made of identical materials. So we will exclude condition 5. b (see the beginning of this section) from our considerations.

If a prime (′) denotes differentiation with respect to x and a dot (·) differentiation with respect to time t, we arrive from (7), (9) and (2) to the following statement of the problem:

$$\mathbf{v} = (v_x, v_y);$$

lower case nu.

$$v_x = V\upsilon_x - V\phi y + VS_x(X' - \dot{X}/V) \quad \text{(a)}$$

lower case nu.

$$v_y = V\upsilon_y + V\phi x + VS_y(Y' - \dot{Y}/V) \quad \text{(b)}$$

$$\mathbf{s}(s_x, s_y) = \mathbf{v}/V; \text{ relative slip;} \quad \text{(c)} \qquad (10)$$

$$v_x = v_y = 0 \rightarrow \|(X, Y)\| \leq fZ \quad \text{(d)}$$

adhesion area

$$\mathbf{v} \neq 0 \rightarrow (X, Y) = fZ\mathbf{v}/\|\mathbf{v}\| \quad \text{(e)}$$

slip area

$$' : \frac{\partial}{\partial x}; \quad \dot{} : \frac{\partial}{\partial t} \quad \text{(f)}$$

Linearized theory

One of the great difficulties in the analysis of rolling contact is the determination of the area of adhesion, where the slip vanishes, and the area of slip. Hence it was proposed by de Pater[3] to treat the case in which the area of slip is so small that its influence can be neglected. This approach was elaborated by Kalker in Refs. 2 and 4. These theories are steady-state theories in which the time derivatives $(\dot{X}, \dot{Y})$ vanish. Also, it is assumed that $v_x = v_y = 0$ everywhere in the contact area, but the restriction $\|(X, Y)\| \leq fZ$ is dropped. The equations are:

$$\begin{aligned} 0 &= \upsilon_x - \phi y + S_x X' \rightarrow \\ X &= -(\upsilon_x - \phi y)x/S_x + f(y) \\ 0 &= \upsilon_y + \phi x + S_y Y' \rightarrow \\ Y &= -(\upsilon_y + \tfrac{1}{2}\phi x)x/S_y + g(y) \end{aligned} \qquad (11)$$

It is seen that two arbitrary functions $f(y)$ and $g(y)$ occur in (11).

Exactly the same happens in the theory of de Pater-Kalker, and f and g are determined on the ground of the same consideration in both theories, as follows.

It is observed that at the leading edge particles come into contact as they enter the contact area. At that moment, they carry no traction. The particles penetrate the contact area along a line parallel to the rolling direction (x-axis), and as a consequence of the no-slip condition and the fact that creepage and spin do not vanish, traction builds up. Finally the particles leave the contact area, whereupon suddenly the traction falls to zero. From this argument it is clear that we must demand that the traction is continuous at the leading edge; more specifically, the traction must vanish at the leading edge. So, X and Y become

$$\begin{aligned} X &= (\upsilon_x - \phi y)\{L(y) - x\}/S_x \\ Y &= [\upsilon_y\{L(y) - x\} + \tfrac{1}{2}\phi\{L(y)^2 - x^2\}]/S_y \end{aligned} \qquad (12)$$

$L(y)$ = coordinate of leading edge, see Fig. 1 and Eq. (1) $= a\sqrt{(1 - y^2/b^2)}$.

X and Y may be integrated over the contact area C, to yield the total force components F_x and F_y, and the torsional moment M_z about the axis of Z which passes through the centre of the contact area. They are compared with the expressions for F_x, F_y, M_z of the exact theory: See [2] p. 90

$$\begin{aligned} F_x &= \iint_C X\,dxdy = 8a^2 b\upsilon_x/(3S_x) && \text{(a)} \\ &= GabC_{11}\upsilon_x \\ F_y &= \iint_C Y\,dxdy \\ &= 8a^2 b\upsilon_y/(3S_y) + \pi a^3 b\phi/(4S_y) && \text{(b)} \\ &= Gab[C_{22}\upsilon_y + \sqrt{ab}C_{23}\phi] \\ M_z &= \iint_C (xY - yX)\,dxdy \\ &= -\pi a^3 b\upsilon_y/(4S_y) + 8a^2 b^3\phi/(15S_x) && \text{(c)} \\ &\doteq G(ab)^{3/2}[C_{32}\upsilon_y + C_{33}\sqrt{ab}\phi] \end{aligned} \qquad (13)$$

C_{ij}: creepage and spin coefficients, tabulated in references 2 and 5

G : modulus of rigidity

where the C_{ij} are the creepage and spin coefficients, which for the exact theory are tabulated in references 2 and 5. Both the exact model and the simplified model predict that F_x depends only on υ_x, and F_y and M_z only on υ_y and ϕ. Also it is seen from (13b) and (13c) that the simplified theory predicts that $C_{23} = -C_{32}$, a relationship which also appears in the exact theory.

Traction bound

We now turn to the discussion of the non-linearised model in which Coulomb's law is fully taken into account. An important role is played by the traction bound fZ, to which this section is devoted. There are in principle three types of traction bound fZ which we will consider.

$$\text{I. } fZ = fZ_0\sqrt{(1-x^2/a^2-y^2/b^2)},\quad f \text{ constant} \tag{14.I}$$

This is the traction bound in accordance with the Hertz theory. However, the x, y derivatives at the edges of the contact area are infinitely large. The rate of increase of the tangential traction is also infinitely large at the edges of the contact area in the exact theory, but it is always finite in the simplified model, see Eq. (10 a, b), (11). Now, the only way in which a state of complete sliding may occur is when the initial slope of the traction bound is smaller in absolute value than de absolute value of the adhesion slope, see Fig. 2. Since this is not possible with traction bound (14.I) we seek an alternative.

$$\text{II. } fZ = fZ_0\{1-x^2/a^2-y^2/b^2\} \tag{14.II}$$

This traction bound is simple, but leads to numerically inaccurate results. It is used in the discussion of the theory of steady state rolling with pure creepage of the following section. In the discussion of steady state rolling with combined creepage and spin the third possibility is used:

$$\begin{aligned}\text{III. } fZ &= fZ_0 A(y)\{1-x^2/a^2-y^2/b^2\} \\ &\quad \text{if } |x| \geq 0.9a\sqrt{(1-y^2/b^2)} = 0.9L(y) \\ &= fZ_0\{\sqrt{(1-x^2/a^2-y^2/b^2)}+ \\ &\quad +B(y)\} \\ &\quad \text{if } |x| \leq 0.9a\sqrt{(1-y^2/b^2)} = 0.9L(y)\end{aligned} \tag{14.III}$$

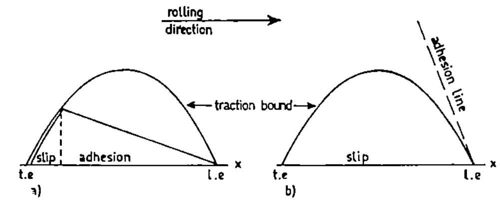

Fig. 2. Traction due to pure creepage: a) *partia slip*, b) *complete slip.*

where $A(y)$, $B(y)$ are determined by the demand that at $x = 0.9L(y)fZ$ is continuous and continuously once differentiable.

The traction bound (14.III) has the advantage of having a finite slope at the edge of the contact area, and is sufficiently like the exact traction bound (14.I) to yield numerically good results. $A(y) = 0.5(1-y^2/b^2)^{-1/2}(1-.9^2)^{-1/2}$

$B(y) = -0.5(1-y^2/b^2)^{1/2}(1-.9^2)^{1/2}$

Steady-state rolling with pure creepage

In the present section the case of rolling with pure longitudinal creepage ($\upsilon_y = \phi = 0$) is considered. Pure lateral creepage ($\upsilon_x = \phi = 0$) is completely analogous as is the case of pure creepage ($\phi = 0$) when, at any rate, the weaknesses S_x and S_y are equal.

When $\upsilon_y = \phi = 0$, the lateral traction Y vanishes, and the problem reads

$$\begin{aligned} s_x &= \upsilon_x + S_x X', \\ |X| &\leq fZ_0\{1-x^2/a^2-y^2/b^2\} \\ s_x &\neq 0 \rightarrow X = \\ &= fZ_0\{1-x^2/a^2-y^2/b^2\} \text{ sign }(s_x)\end{aligned} \tag{15}$$

where the traction bound (14.II) has been adopted. Similar expressions can be given if traction bound (14.I) or (14.III) is derived.

Let $\upsilon_x \geq 0$. At the leading edge, $X = 0$, and the particles tend to adhere, hence

$$X' = -\upsilon_x/S_x \rightarrow X = \upsilon_x\{L(y)-x\}/S_x \quad \text{(tentatively set)} \tag{16}$$

The traction bound has the slope at the leading edge

$$\begin{aligned} fZ' &= -2fZ_0 x/a^2 = \\ &= -2fZ_0 L(y)/a^2 = \\ &= -2f(Z_0/a)\sqrt{(1-y^2/b^2)}\end{aligned} \tag{17}$$

On the leading edge there are two possibilities, either adhesion or slip:

$$\begin{aligned} &2f(Z_0/a)\sqrt{(1-y^2/b^2)} \geq |\upsilon_x|/S_x \\ &\text{adhesion at leading edge} \\ &f(Z_0/a)\sqrt{(1-y^2/b^2)} \leq |\upsilon_x|/S_x \\ &\text{slip at leading edge}\end{aligned} \tag{18}$$

see Fig. 2. Assume that υ_x and y are so that adhesion occurs. According to (16), X increases with decreasing x, until at a certain point the traction bound is reached, see Fig. 2. For still smaller x, there will be slip, for, as seen from (15) and Fig. 2,

$$s_x = \upsilon_x + S_x X' \geq \upsilon_x + S_x(-\upsilon_x/S_x) = 0\,,\quad X \geq 0\,.(\upsilon_x > 0)$$

$$s_x = \upsilon_x + S_x X' \leq \upsilon_x + S_x(-\upsilon_x/S_x) = 0,\quad X \leq 0\,.(\upsilon_x < 0) \tag{19}$$

We determine the boundary between slip and adhesion.

$$\begin{aligned}\{\upsilon_x(L(y)-x)\}/S_x &= \\ &= fZ_0\{1-y^2/b^2-x^2/a^2\} = \\ &= fZ_0\{L(y)^2-x^2\}/a^2 \rightarrow x = \\ &= -L(y)+a^2\upsilon_x/(fZ_0S_x)\end{aligned} \tag{20}$$

from which it appears that the slip-stick boundary is the trailing edge, shifted over a distance $a^2\upsilon_x/(fZ_0S_x)$. The form is shown in Fig. 3; it is in complete accordance (except for the numerical value of the trailing edge shift) with the findings of Haines and Ollerton[6]. It should be noted that this form of the area of adhesion can only be obtained with traction bound (14.II).
The total force can be computed. It is:

traction bound: (14.II),

$$|X| \leq \mu Z_0(1-x^2/a^2-y^2/b^2);$$

$$\begin{aligned}F_x = \iint_C X\,dx\,dy &= \\ &= \mu Z_0(ab/3)\{3 \arcsin \delta - \\ &\quad -12\delta^2 \arccos \delta + \\ &\quad +(13\delta+2\delta^3)\sqrt{(1-\delta^2)}\},\\ \delta &= \upsilon_x a/(2fZ_0S_x) \qquad f = \mu\end{aligned} \tag{21}$$

Combined creepage and spin: a numerical method

In sec. 1, the phenomena in the adhesion area are described. In sec. 2, the slip area is considered. In sec. 3, the leading edge is considered. In sec. 4, final observations are made. The motion is assumed to be in a steady state, so that $X^{\cdot} = Y^{\cdot} = 0$.

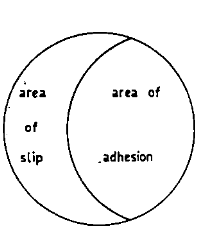

Fig. 3. Contact area distribution for pure creepage.

1. The phenomena in the adhesion area

In an area of adhesion, the following equations hold:

$$\begin{aligned}0 = s_x &\equiv \upsilon_x - \phi y + S_x X' \\ 0 = s_y &\equiv \upsilon_y + \phi x + S_y Y'\end{aligned} \tag{22}$$

if the position where a particle enters this particular area of adhesion is denoted by (x_a, y) and the traction in that point by (X_a, Y_a), Eq. (22) yields (see also (11))

$$\begin{aligned}X &= 1/S_x\,.(\upsilon_x-\phi y)(x_a-x)+X_a \\ Y &= 1/S_y\,.\{\upsilon_y(x_a-x)+ \\ &\quad +\tfrac{1}{2}\phi(x_a^2-x^2)\}+Y_a\,.\end{aligned} \tag{23}$$

The adhesion area extends backwards, along a line parallel to the x-axis from x_a to the point where again $X^2+Y^2 = f^2Z^2$:

$$\text{Adhesion} \rightarrow X^2+Y^2 \leq f^2Z^2 \tag{24}$$

2. The phenomena in the slip area

The following equations hold in the area of slip:

$$X = fZ\cos\theta\,,\quad Y = fZ\sin\theta; \tag{25a}$$

$$\begin{aligned}s_x &\equiv \upsilon_x-\phi y+S_xX' = \lambda X \\ s_y &\equiv \upsilon_y+\phi x+S_yY' = \lambda Y,\quad \lambda > \theta\end{aligned} \tag{25b}$$

Differentiate (25a) with respect to x:

$$\begin{aligned}X' &= fZ'\cos\theta - Z\theta'\sin\theta; \\ Y' &= fZ'\sin\theta + fZ\theta'\cos\theta.\end{aligned} \tag{26}$$

so that (25b) becomes

$$\begin{aligned}&\upsilon_x-\phi y+S_x(fZ'\cos\theta-fZ\theta'\sin\theta) = \\ &= \lambda fZ\cos\theta \\ &\upsilon_y+\phi x+S_y(fZ'\sin\theta+fZ\theta'\cos\theta) = \\ &= \lambda fZ\sin\theta\end{aligned} \tag{27}$$

From the Eq. (27), λ may be eliminated,

$$\begin{aligned}&fZ\theta'(S_x\sin^2\theta+S_y\cos^2\theta) = \\ &= (\upsilon_x-\phi y)\sin\theta-(\upsilon_y+\phi x)\cos\theta + \\ &\quad +fZ'(S_x-S_y)\cos\theta\sin\theta\end{aligned} \tag{28}$$

f=μ

This is an ordinary differential equation for θ when $\mu Z \neq 0$. When $\mu Z = 0$, that is, at the edges of the contact area, the equation is singular and special measures must be taken which are described in 'conditions at the leading edge' (see below). When $Z \neq 0$, the equation can be solved numerically, e.g. by Heun's method. When (SfZ/a) is small in comparison with υ_x, υ_y and ϕ, the numerical integration method tends to be unstable. The instability may be reduced by taking smaller integration steps. However, $\mu Z/a$ is small only in a limited region near the edges of the contact area, so that the instability, when it occurs, has only a very limited effect on the total force.
The condition that $\lambda > 0$ has not been verified. When λ becomes negative, the slip tends to be opposite the traction, and adhesion sets in. We compute λ. To that end multiply the upper Eq. (25b) by $S_y X$ and the lower by $S_x Y$, and add. Then, remembering (25a), we find

$$\lambda f^2 Z^2 (S_y \cos^2\theta + S_x \sin^2\theta) =$$
$$= fZ\{S_y(\upsilon_x - \phi y)\cos\theta + S_x(\upsilon_y + \phi x)\sin\theta\} + S_x S_y (XX' + YY') =$$
$$= fZ\{S_y(\upsilon_x - \phi y)\cos\theta + S_x(\upsilon_y + \phi x)\sin\theta\} + S_x S_y f^2 ZZ';$$

hence

$$\lambda = \{S_y(\upsilon_x - \phi y)\cos\theta + S_x(\upsilon_y + \phi x)\sin\theta + S_x S_y f Z'\}\cdot\{fZ(S_x \sin^2\theta + S_y\cos^2\theta)\}_-{}^{1} \quad (29)$$

The auxiliary condition of slip reads, since $(S_x, S_y) > (0, 0)$:

$$H \equiv S_y(\upsilon_x - \phi y)\cos\theta + S_x(\upsilon_y + \phi x)\sin\theta + S_x S_y f Z' > 0 \quad (30)$$

Adhesion starts when H becomes negative.
It can be shown that when H becomes negative, adhesion may start with $(X^2 + Y^2)$ falling below $f^2 Z^2$, while when $X^2 + Y^2$ starts to exceed $f^2 Z^2$, H becomes positive.

3. Conditions at the leading edge
First must be determined whether there will be slip on the leading edge, or adhesion. Since $Z = 0$ on the leading edge, the condition of adhesion reads

$$(X')^2 + (Y')^2 \leq f^2 (Z')^2,$$

X', Y' determined by (22); Z' by (14)

Insertion of (22) into this equation yields

$$\{-(\upsilon_x - \phi y)/S_x\}^2 + \{-(\upsilon_y + \phi x)/S_y\}^2 \leq \leq f^2 (Z')^2 \rightarrow \text{adhesion} \quad (31)$$

When (31) is satisfied, there will be adhesion on the leading edge, and no complication arises. When (31) is not satisfied, there will be slip on the leading edge, and Eq. (28) is singular.
The first problem is to determine a starting value θ_0 of the angle θ.
If θ' is to be finite, we must have

$$(\upsilon_x - \phi y)\sin\theta_0 - (\upsilon_y + \phi x)\cos\theta_0 + \mu Z'(S_x - S_y)\cos\theta_0 \sin\theta_0 = 0 \quad (32)$$

This may be written as a fourth degree equation in $\sin\theta_0$, which, as such, is impracticable to solve exactly. Instead, (31) is solved by Newton's method. We also need a starting value θ'_0 of θ', in order to be able to move away from the leading edge. To that end, (28) is differentiated, while it is kept in mind that $\mu Z = 0$ and θ''_0 is finite.

4. Final observations on the method
A program was written implementing the method of this section. It was found that it performed well for small and medium values of the creepage and the spin. For large values some difficulties were encountered which took the form of a rapid unrealistic alternation of areas of slip and adhesion. The reason for this is, in our opinion, the singular character of the differential Eq. (28), and in all cases encountered by us it could be remedied by taking smaller x-steps in the integration of Eq. (24), so that the transit from leading to trailing edge takes about 100 steps.

Transient phenomena
Up to now we only considered time-independent, steady state problems, in which $X^{\cdot}$ and $Y^{\cdot}$ could be neglected. We will now consider the simplest case in which this is not so, viz. $\upsilon_y = \phi = 0$, $Y = 0$ while the bodies have the form of two long cylinders with parallel axes, about which they are rotated. The lateral (y) coordinate may be disregarded and the contact area is given by

$$\text{contact area}: |x| \leq a. \quad (33)$$

Also we will assume complete adhesion as in sec. 3. Under those circumstances the exact model has been treated by Kalker in Ref. 7. The governing equation is

$$0 = s_x \equiv \upsilon(t) + S_x\{X' - X^{\cdot}/V(t)\},$$

$$|x| \leq a; \quad X = 0, \quad |x| > a \tag{34}$$

This is a partial differential equation of the first order for X. The rolling velocity $V(t)$ is independent of the x-coordinate and we can write

$$\frac{1}{V}\frac{X}{\partial t} = \frac{\partial X}{\partial l}; l = \int_{t_0}^{t} V(q)dq =$$

$$= \text{distance traversed} \tag{35}$$

Hence forward, we will replace the time t by the distance traversed, and we again denote by ($^{\cdot}$) differentiation with respect to l. (34) becomes

$$0 = \upsilon(l) + S_x\{X' - X^{\cdot}\},$$

$$|x| \leq a \quad ' : \frac{\partial}{\partial x}; \quad {}^{\cdot} : \frac{\partial}{\partial l}$$

$$X = 0, |x| > a \tag{36}$$

This equation is readily solved:

$$X(x, l) = X(x + l - l_0, l_0) + \int_{l_0}^{l} \frac{\upsilon(q)}{S_x} \quad \text{if } x + l - l_0 \leq a \tag{37a}$$

$$= \int_{x+l-a}^{l} \frac{\upsilon(q)}{S_x} dq \quad \text{if } x + l - a \geq l_0 \tag{37b}$$

It is seen that when $\upsilon(q) = \upsilon$ is constant from the distance l_0 onward, and $l - 2a \geq l_0$, then $X(x, l) = \upsilon(a-x)/S_x$ by (37b), the steady state of sec. 3, independent of l, and independent of the initial traction distribution $X(x, l_0)$. The condition $l - 2a \geq l_0$ signifies that transience is completed after a contact width $2a$ has been traversed, a conclusion which is approximately valid in the exact theory of Ref. 7.

An important traction distribution is that due to a shift without rolling, parallel to the x-axis, of one body with respect to the other. It is called the Mindlin shift and it is described in Ref. 8. The displacement and traction due to it are given by

$$u = \delta = S_x X \rightarrow X = \delta/S_x$$

$$\text{if } |x| < a, \; X = 0 \text{ if } |x| > a. \tag{38}$$

We start rolling at the distance $l_0 = 0$ with a constant creepage υ; according to (37) and (38)

$$X(x, l) = (\delta/S_x) + \int_0^l \upsilon/S_x dq =$$

$$= (\delta/S_x) + \upsilon l/S_x \quad \text{if } x + l \leq a, x \leq a - l; |x| < a$$

$$= \upsilon(a-x)/S_x \quad \text{if } x \geq a - l, |x| < a \tag{39}$$

$$= 0 \quad |x| > a$$

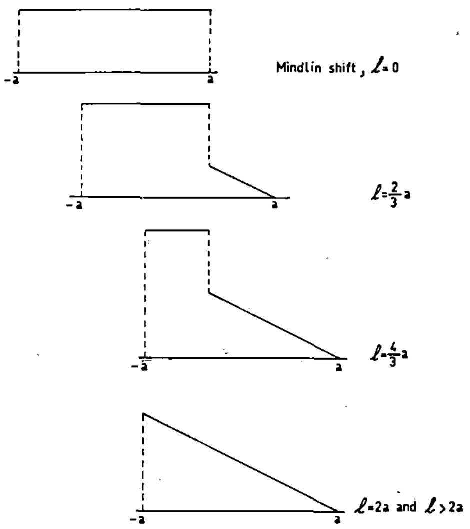

Fig. 4. Transient rolling phenomena.

A few stages of the development of the traction are shown in Fig. 4.

Rolling of bodies with unequal elastic constants

Up to now we have not succeeded in incorporating in the simplified model devices by which can be reproduced the salient features of the phenomena occurring when two bodies with different elastic constants roll over each other.

Use of the simplified model as a quantitative theory

The qualitative agreement between the simplified and the exact theories is so striking, that the question arises whether by a proper choice of the parameters of the problem we can get an approximate quantitative agreement. It appears that this is indeed so for steady-state rolling of elastically symmetric bodies, see section 'combined creepage and spin'.

I. The traction bound
It is advisable to use the traction bound (14.III):

$$fZ = fZ_0 A(y)\{1-x^2/a^2-y^2/b^2\} \quad \text{if } |x| \geq 0.9a\sqrt{(1-y^2/b^2)}$$
$$= fZ_0\{\sqrt{(1-x^2/a^2-y^2/b^2)}+B(y)\} \quad \text{if } |x| \leq 0.9a\sqrt{(1-y^2/b^2)} \qquad (14.\text{III})$$

where $A(y)$ and $B(y)$ are determined by the demand that the traction bound is continuous and continuously differentiable at $|x| = 0.9a\sqrt{(1-y^2/b^2)}$. Z_0 is determined by the demand that

$$\iint_C fZ\,dxdy = fN, \; N\text{: total normal force.} \qquad (40)$$

II. The coincidence of the creepage and spin coefficients
It seems reasonable to demand that the initial slopes of the $(F_x, F_y, M_z)/(\upsilon_x, \upsilon_y, \phi)$ diagrams should be coincident in the simplified theory and in the exact theory, that is, the creepage and spin coefficients C_{ij} in both theories should coincide. The exact creepage and spin coefficients have been tabulated in Ref. 2 and according to (13) we must have

$$C_{11} = 8a/(3S_xG);$$
$$C_{22} = 8a/(3S_yG);$$
$$C_{33} = 8b/(15S_xG);$$
$$C_{23} = -C_{32} = \pi a^{3/2}/(4b^{1/2}S_yG). \qquad (41)$$

We have only 2 parameters, viz. S_x and S_y, and 5 equations to be met.
As solution out of this difficulty we propose that a separation is made between the calculation of the moment M_z and the calculation of the forces F_x, F_y. As to the moment, we have the equations

$$C_{33} = 8b/15S_xG \rightarrow S_x = 8b/15C_{33}G;$$
$$C_{32} = -\pi a^{3/2}/(4b^{1/2}S_yG) \rightarrow S_y = \qquad (42)$$
$$= -\pi a^{3/2}/(4b^{1/2}C_{32}G)(C_{32}<0)$$

moment calculation

and S_x and S_y are un ambiguously determined. As to the forces, we have the equations

$$C_{11} = 8a/(3S_xG) \rightarrow S_x = 8a/(3C_{11}G)$$
$$C_{22} = 8a/(3S_yG) \rightarrow S_y = 8a/(3C_{22}G)$$
$$C_{23} = \pi a^{3/2}/(4b^{1/2}S_yG) \rightarrow S_y =$$
$$= \pi a^{3/2}/(4b^{1/2}C_{23}G). \qquad (43)$$

Here, we have two different definitions of S_y. So we propose to enter our programme with spin ϕ, and to calculate internally with $h\phi$; then

$$\left.\begin{aligned} C_{23} &= (\pi a^{3/2}/[4b^{1/2}S_yG])h \\ S_y &= 8a/(3C_{22}G) \end{aligned}\right\} \rightarrow$$
$$h = \frac{32}{3\pi}\sqrt{\left(\frac{b}{a}\right)}\frac{C_{23}}{C_{22}}. \qquad (44)$$

The exact and simplified theories are compared in Fig. 5 and Fig. 6. It is seen that the coincidence may be termed reasonable and probably is sufficient for most needs.
Finally the simplified theory may be used directly for the case that the bodies are covered with a thin elastic layer which responds to shear in the following manner:

$$u_l = L_xX, \quad v_l = L_yY. \qquad (45)$$

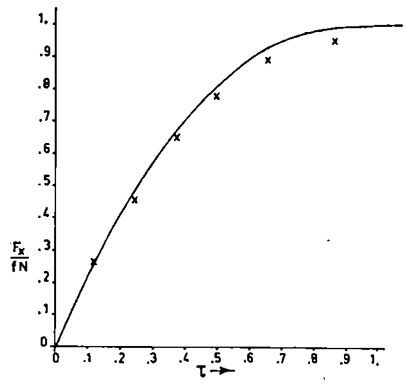

Fig. 5. A comparison between Kalker's empirical formula of Ref. 5 and the simplified theory. τ: a creepage parameter. Drawn: empirical formula; x: simplified theory.

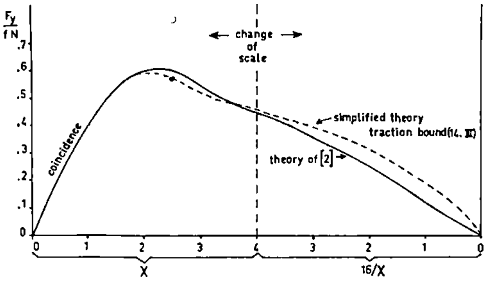

Fig. 6. Pure spin, $v_x = v_y = 0$. χ: a spin parameter. Simplified theory and theory of Ref. 2. (page 124)

Then in the moment calculations one must use

$$S_x = \{8b/15\,C_{33}\,G\} + L_x,$$

$$S_y = \{-\pi a^{3/2}/4b^{1/2}\,C_{32}\,G\} + L_y \qquad (46)$$

moment calculation.

and in the force calculation:

$$S_x = \{8a/3\,C_{11}\,G\} + L_x,$$

$$S_y = \{8a/3\,C_{22}\,G\} + L_y,$$

$$h = \{L_y + S_{y0}\}/\{L_y + S_{y0}/h_0\},$$

$$S_{y0} = S_y(43) = 8a/3\,C_{22}\,G, \qquad (47)$$

$$h_0 = h(44) = \frac{32}{3\pi}\sqrt{b/a}\,\frac{C_{23}}{C_{22}}$$

force calculation.

It is to be expected that the agreement between simplified theory and exact theory tends to improve as L_x and L_y become larger.
At this point we would like to remark that the drastic decrease of the creepage coefficient below the value predicted by the exact theory which is described by Hobbs[9], may be due to an elastic layer covering the wheel and the rail which is weaker in x-direction than in y-direction ($L_x > L_y > 0$).
It is also possible to investigate layers with non-elastic response to shear by means of the simplified theory, but we will not investigate that further.

Conclusion

It has been shown to be possible drastically to simplify the equation of elasticity and still reproduce all salient features of rolling contact phenomena with the exception of those which are a consequence of the elastic asymmetry of the bodies. Also, the simplified theory may be used as a quantitative approximation of the exact theory. Calculations with the simplified theory are about 100 times faster than with the three-dimensional exact theory.
Finally, the presence of an elastic layer on the bodies may be taken into account, and it is proposed that the discrepancies between railway experiments and theory may be due to just such a layer.

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