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A Physically Based Method of Enhancement of Experimental Data–Concepts, Formulation, and Application to Identification of Residual Stresses

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Summary

This paper presents the formulation for a new approach to enhancing experimental data. Using simultaneously all the information available for the problem being studied, i.e, physically based approximation, this approach offers potential advantages over the typical methods used for fitting a curve or a surface to experimental data. Those methods are based primarily on the least squares technique or on a similar mathematical method. While their results are usually compared with a relevant theoretical solution, the new combined approach produces results simultaneously from both the various experimental measurements and the theoretical model. In this way only one, i.e., "the best," solution is obtained. The new approach tries to fit best the experimental data and, at the same time, to satisfy, as much as possible, the requirements of the theory involved.

In terms of mathematics, the new approach yields constrained optimization problems for a functional composed of experimental and theoretical parts.

Presented are the general concept of the approach and, as a particular example, its formulation for strain gage measurements, as well as its application for residual stress analysis.

Contents

Introduction

Following some earlier ideas [1-8], this paper presents a new approach to enhancing experimental data. It uses simultaneously all information available for the problem being considered. The approximation obtained in this way is physically based, since all basic physical relations relevant to the problem are taken into account. Moreover, the results obtained are within the error bounds corresponding to the appropriate physical statistics.

This approach is especially convenient when data are insufficient, uncertain, not uniformly distributed, or missing, e.g., in regions where measurements are often difficult or even impossible. In this way, solving ill-conditioned problems can be avoided, and better quality results can be obtained. The fitted solution is smooth enough and partially cancels experimental data errors. Therefore, its differentiation yields, if required, reasonable effects.

In many engineering problems, a multidimensional physical field (e.g., stress in mechanics) is needed at each point of the domain considered, but only a function of the required quantities (e.g., isochromatics) can be measured. Such information is often insufficient to determine the unknown field. However, taking into account appropriate physical relations that ought to be fulfilled by an unknown field, it is usually possible to obtain all required information.

The main objective of this paper is to present several variants of the general formulation of a problem using this new approach to enhancing the data. Stress analysis is used as a particular case. Examples of application to enhancing strain gauge measurements and moire interferometry data are also presented.

Formulation

Formulations based on two concepts of analysis are considered: (1) global and (2) global-local. Both of these concepts are posed as the constrained optimization problems. Using global analysis, enhanced results are obtained simultaneously in the whole domain. Global-local analysis also uses measured data from the entire domain, but it provides results only in one required point at a time.

Global Method

All information available about a problem can be used. The problem is posed in the following general way:

Find the stationary point of the functional

$$\Phi = \lambda \Phi^E + (1-\lambda)\Phi^T, \quad \lambda \in [0,1] \tag{1}$$

satisfying the equality constraints (theoretical nature)

$$A(\sigma) = 0 \tag{2}$$

and the inequality constraints (usually experimental nature)

$$B(\sigma) \leq e \tag{3}$$

Here, $\Phi^T(\sigma)$ and $\Phi^E(\sigma)$ are the theoretical and experimental parts of the functional, σ is the required solution, and λ is a scalar weighting factor.

Experimental Requirements

Functional

An averaged global "error norm" is introduced as the experimental part of the functional, as follows:

$$\Phi^E(\sigma) = \frac{1}{N} \sum_{n=1}^{N} F \left[\frac{f(\sigma(r_n)) - f_n^{exp}}{e_n} \right], \tag{4}$$

where σ represents the required unknown field, f is a measured function of σ, f_n^{exp} is its experimental value at the point f_n, e_n is an admissible experimental error, N is a number of measurements, $F(x) = p(\bar{x}) - p(x-\bar{x})$ is a data scattering function defined by the probability density function $p(x-\bar{x})$, and $\bar{x}$ is the expected value.

The results of various kinds of experiments, e.g., photoelasticity, moire interferometry, and the strain gauge technique, can be combined to determine the averaged global error:

$$\Phi^E = \frac{1}{I} \sum_{i=1}^{I} \Phi_i^E \tag{5}$$

The summation can be extended to all types of experiments, because the error norm (4) is dimensionless and particular types of measurements are relatively normalized by e_n, which stands for a weighting factor. More precise measurements result in a smaller e_n. Consequently, the share of such experiment in an error function increases.

Constraints

The enhanced field $\sigma(r)$ cannot differ too much from experimental data. Thus, the constraints $B(\sigma) \leq 0$ are defined as local requirements:

$$|f(\sigma(r_n)) - f_n| \leq e_n, \qquad n=1,2,...,N \tag{6}$$

Moreover, it is useful to also impose an averaged global constraint:

$$\Phi^E \leq e_E \tag{7}$$

In practical analysis, both types of constraints, (6) and (7), or only one of them can be used. Admissible experimental errors e_E and e_n , $n = 1,2,\dots,N$ should be evaluated taking into account the true statistics of measurements.

Theoretical Requirements

Functional

Concerning the theoretical part of the functional Φ^T , two situations can be distinguished:

(i) *When the theory is known* - In mechanics, $\Phi^T(\sigma)$ can be represented by a well known energy functional that has to be minimized, e.g., the total complementary energy of statically admissible stresses.

(ii) *When the theory is not known* - A heuristic principle, e.g., requirement of smoothness, i.e., minimal average curvature $\kappa(\sigma)$, can be used.

For a scalar function f, a mean value of the second directional derivative can be assumed, and the following definition of its average curvature at the given point can be introduced:

$$\kappa^2 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial v^2}\right)^2 d\varphi \tag{8}$$

This definition is objective with the respect to any rotation of the coordinate system, and can be extended to the case of a tensorial function σ_{ij} (e.g., stresses) to obtain

$$\kappa^2(\sigma_{ij}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^2\sigma_{ij}}{\partial v^2}\,\frac{\partial^2\sigma_{ij}}{\partial v^2}\, d\varphi \tag{9}$$

In the Cartesian coordinate system, these definitions can be expressed respectively as

$$\kappa^2(f) = \frac{1}{4}(f_{,xx} + f_{,yy})^2 + \frac{1}{8}(f_{,xx} - f_{,yy})^2 + \frac{1}{2}f^2_{,xy}$$

$$\kappa^2(\sigma_{ij}) = \frac{3}{8}(\sigma_{ij,xx}\,\sigma_{ij,xx} + \sigma_{ij,yy}\,\sigma_{ij,yy}) + \frac{1}{4}\sigma_{ij,xx}\,\sigma_{ij,yy} + \frac{1}{2}\sigma^2_{ij,xy}\ .$$

The theoretical (heuristic) part of the functional can then be assumed to be in the following form:

$$\Phi^T = \frac{1}{\Omega} \int_\Omega \kappa^2 \, d\Omega \qquad (10)$$

Constraints

Theoretical constraints are usually presented as equality conditions $A(\sigma) = 0$. For example, for the complementary energy functional, these are the equilibrium equations $\sigma_{ij,j} = 0$ in Ω and the static boundary conditions $\nu_i \sigma_{ij} = p_i$.

Specific Formulations Proposed

One of the main disadvantages of the general formulation (1)-(3) is the problem of how to establish the weighting factor λ , i.e., how to determine a reasonable balance between experiment and theory. The following specific formulations have been proposed to address this problem:

(i) Experimental requirements are removed from the functional Φ and appear only as constraints (6) and (7). The objective is to find

$$\min_\sigma \Phi^T \qquad (11)$$

satisfying the constraints

$$A(\sigma) = 0 \,, \quad \Phi^E \leq e_E \,, \quad |f(\sigma(r_n)) - f_n^{exp}| \leq e_n \,, \quad n = 1,2...,N \qquad (12)$$

(ii) The problem (1)-(3) is divided into two steps expressed as subsequent optimization problems. The family of admissible solutions $\sigma(\lambda)$ is evaluated in step 1, and the optimal value of λ is determined in step 2.

Step 1: Find $\sigma(\lambda)$ that yields the stationary value of the functional

$$\Phi = \lambda \Phi^E + (1-\lambda)\Phi^T \qquad (13)$$

satisfying the theoretical equality constraints $A(\sigma) = 0$.

Step 2: Find the minimal value of λ

$$\min_\lambda \lambda \qquad (14)$$

satisfying admissible bounds of global and local experimental errors

$$\Phi^E \leq e_E , \quad |f(\sigma(r_n))-f_n^{exp}| \leq e_n , \quad n = 1,2,...,N . \tag{15}$$

(iii) The original formulation (1)-(3) is used, but with λ considered as variable throughout the domain. A special technique of local λ adjustment was proposed in [2,7], but will not be discussed here.

Each of these specific formulations contains a constrained optimization problem. However, the second one (ii) seems to be computationally the most effective, especially when the constraints $A(\sigma) = 0$ are linear.

Global-Local Method

This method involves searching for a local surface (curve) that in a neighborhood of the considered points fits the required solution the best. The concept of the method is based on the local expansion of the following searched field function into the Taylor series of the order p at a considered point r_k:

$$\sigma(r-r_k) = \sum_{i,j=o}^{i+j\leq p} c_{ij} (x-x_k)^i (y-y_k)^j \tag{16}$$

Unknown values of c_{ij} are found by minimizing of the global error Φ with equality constraints $A(\sigma(c_{ij})) = 0$ locally enforced:

$$\min_{c_{ij}} \Phi \ (r_k,\sigma(c_{ij})) \tag{17}$$

Such a problem has to be solved subsequently at each required point. The inequality constraints $B(\sigma) \leq 0$ still have to be satisfied, but will be considered later.

In the mathematical sense, the global-local method may be viewed as a generalization of some earlier concepts of local approximation [9,10,11]. Some preliminaries of the physically based approach to data handling were performed previously by the authors of this paper [5,6,7,8]. A new, advanced version of the method is presented here.

In this new version, the global error function (4) is considered in a modified form with a weighting factor $v(\rho_n)$ introduced:

$$\Phi(r,\sigma) = \frac{1}{N}\sum_{n=1}^{N} v(\rho_n)\, F\left(\frac{f(\sigma(r_n))-f_n^{exp}}{e_n}\right) \qquad (18)$$

This factor depends on the distance $\rho_n = |r_n - r|$ between an analyzed point r and a data point r_n .

The proposed weighting factor function is of the form

$$v(\rho) = \left(\rho^2 + \frac{g^4}{\rho^2 + g^2}\right)^{-m} \qquad (19)$$

where g is so far a free parameter of approximation that has not yet been determined. Power m is chosen in a way that cuts information from a distance ρ_n when it reaches the approximation error level [1], e.g., $m = 3$ when $\rho = 2$.

The way that v is chosen serves the rule "the longer the distance ρ_n , the smaller the influence of experimental data on the considered value of field function σ." Additionally, the weighting factor (19) provides

$$v'(0) = v''(0) = v'''(0) = 0. \qquad (20)$$

Due to these conditions, fitted field function σ is smooth enough and is not "attracted" by an experimental point.

Parameter g is responsible for accuracy of approximation. When $g = 0$, interpolation is performed. On the other hand, when g is infinite, a polynomial approximation in the entire domain (e.g., quadratic for $\rho = 2$) is obtained. The appropriate finite value of g should be chosen based on the experimental error, to satisfy the constraints $B(\sigma) \leq 0$.

The proposal is to solve the experimental data smoothing problem using the global-local method in two steps. Step 1 is performed to solve the original optimization problem (17) for a given g . Step 2 is performed to determine g . Then such g will be used to find the final solution to the problem. Two options are proposed for determining g :

(i) Find

$$\max_{g} g \tag{21}$$

satisfying the experimental constraints $B(\sigma) \leq 0$, i.e.,

$\Phi(r,\sigma) \leq e_E$, and/or $|f(\sigma(r_n))-f_n^{exp}| \leq e_n$, $n=1,2,\ldots,N$.

(ii) Find

$$\min_{g} \Phi(r,\sigma) \tag{22}$$

In formulation (ii), a solution of the original optimization problem (17) is sought with all experimental points taken into account, excluding, however, the measured value at the considered point.

The criteria satisfying the constraints $|f(\sigma(r_n))-f_n^{exp}| \leq e_n$, $n=1,2,\ldots,N$ are related to the experimental error and are compatible with formulation (1)-(3).

Formulation for Strain Gauge Measurements

In this case, the global error function (18) consists of two terms

$$\Phi = \Phi_{Int} + \alpha\, \Phi_b \tag{23}$$

The first one is given by

$$\Phi_{Int}(r,\sigma) = \frac{1}{N}\sum_{n=1}^{N} v(\rho_n)\, \frac{1}{3}\sum_{j=1}^{3} (\sigma_j(r_n) - \sigma_{j(n)}^{exp})^2 \tag{24}$$

where $\sigma_j = \{\sigma_{xx},\sigma_{xy},\sigma_{yy}\}$, $\sigma_{j(n)}^{exp}$ are "measured stresses" obtained from strain gauge rosettes.

The second is the boundary term:

$$\Phi_b(r,\sigma) = \sum_{l=1}^{L} v_l\, \frac{1}{2}\left[(\sigma_1 \nu_1 + \sigma_3 \nu_2 - q_1)^2 + (\sigma_3 \nu_1 + \sigma_2 \nu_2 - q_2)^2\right] \tag{25}$$

where L is the total number of boundary points, q is loading, and ν is the vector normal to the boundary.

In the global-local formulation considered here, the boundary conditions cannot be satisfied exactly. Therefore, they were moved from the equality constraints to the functional, and a weighting factor $\alpha > 1$ was assumed, to better enforce their fulfillment.

On the other hand, as mentioned above, the equilibrium equations are strictly satisfied in advance. At each point considered, exact satisfaction of the equilibrium equations is enforced in the expansion of stresses into the Taylor series, thereby reducing the number of independent unknown parameters. For example, for the quadratic approximation σ_j, there may be (3 x 6) - (2 x 3) = 12 independent unknowns that are found by minimizing functional (18) from the resulting 12 simultaneous linear equations. Such a procedure may be fast and easily repeated in each subsequent point of the domain.

Application of the Global-Local Method to Evaluation of Residual Stress

For some technical problems (e.g., analysis of railroad rails or wheels under revenue service conditions), information is required about distribution of residual stresses in the body. This is usually a complex problem, requiring use of various experimental methods to find the stresses. Relevant experiments are usually expensive, and do not yield results of sufficient quality. Therefore, enhanced methods for evaluating measured data are being considered, especially when the strain gauge technique or moire interferometry is used.

The approach described in this paper in general terms is verified in numerical experiments where random but controlled distribution of residual stresses is simulated. An example of one of these experiments, where strain gauges were randomly distributed in the circular domain, is shown in Figure 1. In that experiment, true stresses evaluated from an assumed Airy function were subjected to artificial randomization. Results are presented for all three stress components. The true values (the continuous lines) are compared with the simulated experimental values (dots) and with their approximation using the global-local approach (the dashed lines). Three different values of g were used: zero (interpolation), a large value (g = 10) that is sufficient to generate the global quadratic approximation, and the optimal value based on formulations (17) and (22)-(25).

Final Remarks

The new general approach to enhancing experimental measurements is presented in this paper. It uses all available information about the given problem, both experimental and theoretical (or even heuristic) nature. Several particular formulations are proposed, in the form of constrained optimization problems. They reflect two different techniques: global and global-local. The global technique provides a solution at once for the entire considered domain, but it may demand large computer power. The global-local technique yields a solution in one point at a time, and may be subsequently applied at any required point. A test example for strain gauge measurements is included.

The approach has potential for broad application. It seems to be specially promising for enhancing insufficient or uncertain data.

Research is currently being conducted. Further studies are planned, including intensive testing and various applications of the approach, especially in experimental analysis of residual stresses.

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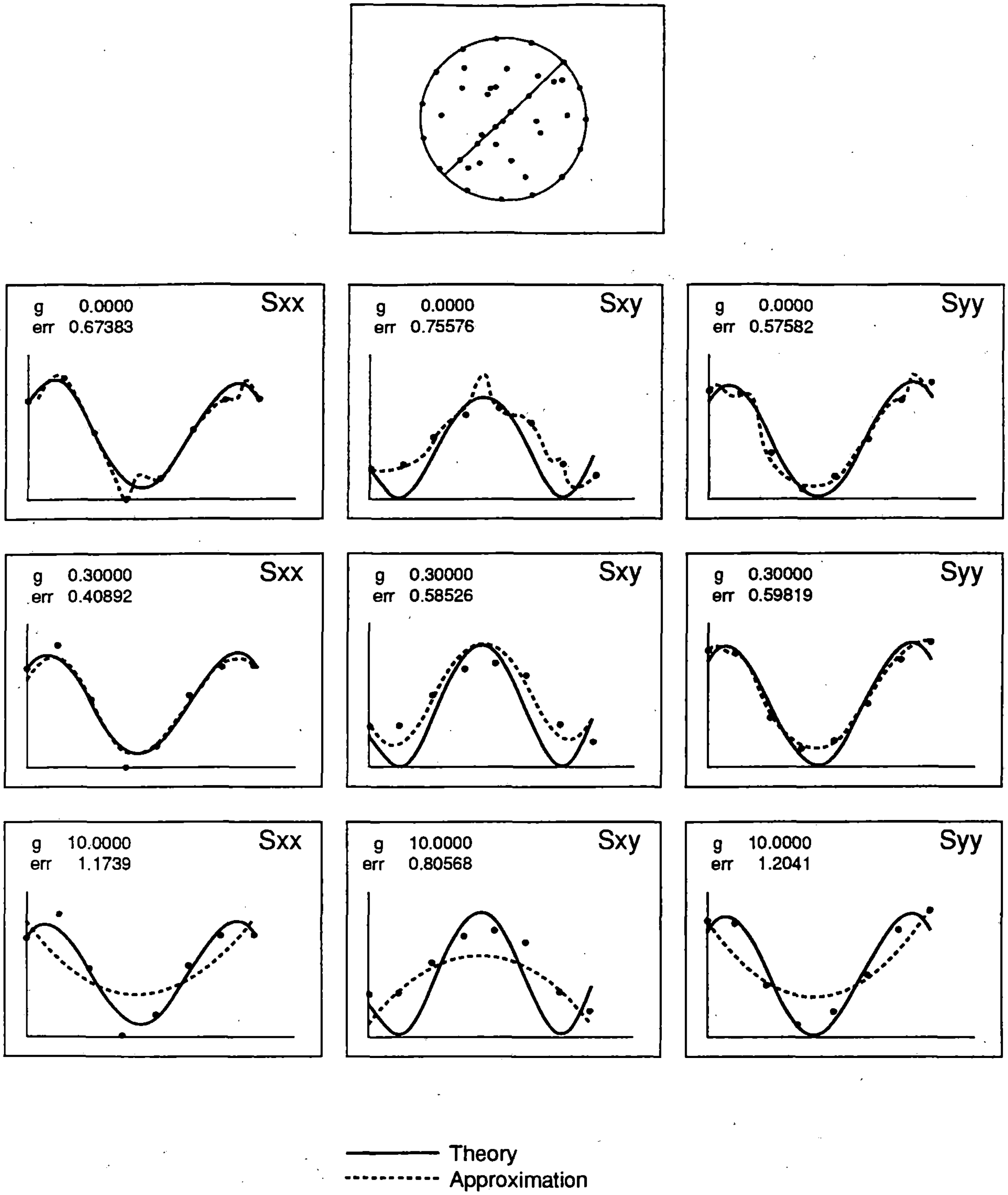

Figure 1. The distribution of rosettes and the results