FYI
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Federal Railroad Administration

One-Dimensional Problem of a Thick-Walled Cylinder Subjected to Internal Pressure and Combined Load Analysis of Plastic and Residual Strains

Study Paper
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Table 1

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Introduction

This report presents plastic and residual strains corresponding to residual stresses obtained for two one-dimensional test problems using a thick-walled cylinder. General formulation of the problems was the same as in the test of torsion of a cylinder, presented in the previous report. The cylinder was is subjected to:

- Internal pressure p
- Combined load (internal pressure p plus torsion m)

Analysis

The analysis consists of two steps:

STEP I: Find self-equilibrated stresses σ_{ij} that minimize the functional

$$\min_{\sigma_{ij}} \int_V \frac{1}{2} C_{ijk1} \, \sigma_{ij} \, \sigma_{k1} \, dV + \int_V \epsilon^p_{ij} \, \sigma_{ij} \, dV$$

$$i, j, k, 1 = 1, 2, 3.$$

and satisfy

$\sigma_{ij,j} = 0$ in V equilibrium equations

$\sigma_{ij} n_j = 0$ on ∂V homogeneous static boundary conditions

The relation $\sigma_{ij} = \sigma_{ij} \; (\epsilon^p_{ij})$ is obtained in this way.

STEP II: Find plastic strains ϵ^p_{ij} that provide the lowest value of the functional

$$\min_{\epsilon^p_{ij}} \int_V \frac{1}{2} C_{ijkl} \, \sigma_{ij} \, (\epsilon^p_{mn}) \, \sigma_{kl} \, (\epsilon^p_{mn}) \; dV$$

and satisfy the yield condition

$$\phi \, [\, \sigma_{ij} \, (\epsilon^p_{ij}) + \sigma^E_{ij} \,] \le 0 \qquad \text{in V.}$$

Data

Inner radius: 2.0

Outer radius: 4.0

Material constants: E = 1.0; ν = 0.5

Internal Pressure - Plastic zone:

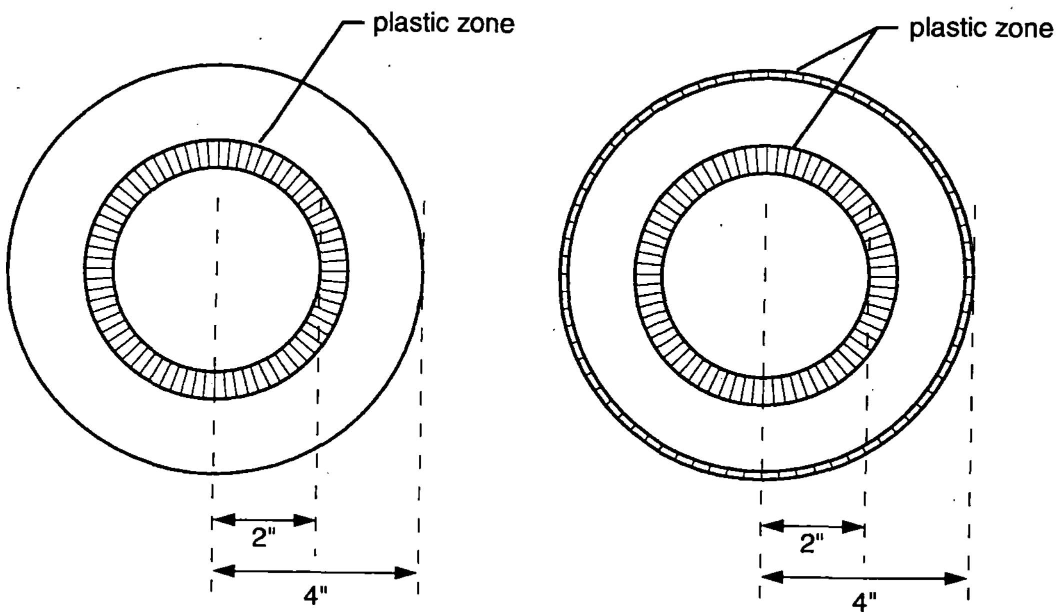
(1) 25% - $2.0 \leq r < 2.5$

(2) 50% - $2.0 \leq r < 3.0$

Combined load - Plastic zone:

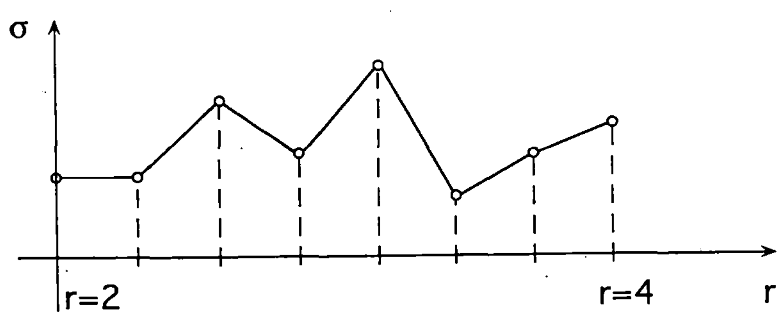
(1) About 50%

(2) About 60%



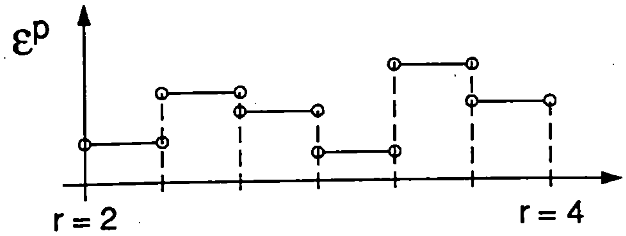
Approximation of Stresses and Strains in Elements

For stresses, the elements were equally spaced piecewise-linear:

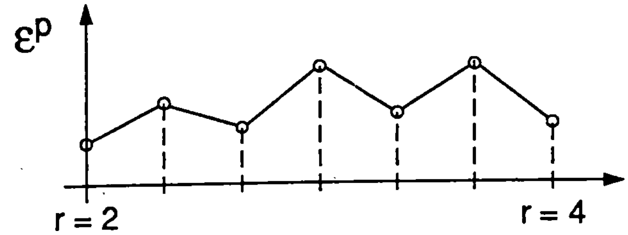


For plastic strains, two different approximations were considered:

(1) Constant



(2) Linear



Both the stresses and the strains were calculated in the centroids of the elements.

Purpose of the Tests

The main goals of these tests were the same as for the test of torsion of the cylinder:

(1) Investigate the quality of the results obtained and the convergence rate as a function of (a) mesh density (i.e., the number of decision variables), and (b) the size of the pre-estimated (overestimated) plastic zone.

(2) Investigate the uniqueness of the residual strains obtained.

Results

The results obtained are shown in Table 1 and in the figures. The results shown in Table 1 and Figures 1 through 15 are from tests for internal pressure. The results shown in Figures 16 through 29 are from tests for combined load. (Note: only convergence could be shown in the tests for combined load because an analytical solution has not been yet found.)

Table 1 summarizes the quality of the results obtained from all of the tests for internal pressure. The maximum number of nodes in the radial direction was 24. The abbreviations used in Table 1 are:

ANAL	-	Analytical quality solution
SAT	-	Satisfactory results
KIN. ADM.	-	A solution obtained that differed from the true solution by the kinematically admissible strain field

The bold vertical lines in Table 1 indicate that no more variables from the elastic zone could be taken, because a divergent solution would be obtained.

Figures 1 through 4 show the results obtained when the estimated (by program) plastic zone was exactly equal to the true plastic zone. These figures show plastic strains, residual stresses, and residual strains for a 50% plastic zone (irregular mesh) that is 50% of the thickness of the cylinder wall. The top (a.) portions of the figures show results for 12 nodes in the plastic zone and 6 nodes in the elastic zone. The bottom (b.) portions show results for 24 nodes in the plastic zone and 12 nodes in the elastic zone. All of these tests showed very good quality results.

The results shown in Figures 5 through 14 are from tests concerning the influence of the overestimated plastic zone on the final solution. These figures clearly show that if the plastic zone that is close to the true zone is taken, the results are good quality. If the plastic zone is close to the full cross-section, i.e., the considered body changes into a mechanism, the final solution can differ from the true solution by the kinematically admissible strain field.

Figures 5 through 8 show plastic strains, residual stresses, and residual strains obtained for a 13-node mesh (a true plastic zone that is 25% of the thickness of the cylinder wall) for the following locations in a pre-estimated plastic zone:

□ 5 elements in the plastic zone

Δ 8 elements in the plastic zone

◇ 10 elements in the plastic zone

x 12 elements in the plastic zone

Figures 9 through 12 show plastic strains, residual stresses, and residual strains obtained for a 15-node mesh (a true plastic zone that is 50% of the thickness of the cylinder wall) for the following locations in a pre-estimated plastic zone:

□ 9 elements in the plastic zone

Δ 11 elements in the plastic zone

◇ 12 elements in the plastic zone

x 14 elements in the plastic zone

All of the results shown in Figures 1 through 12 were obtained when a constant approximation of plastic strains was applied. Figures 13 and 14 show plastic strains and residual strains obtained for linear approximation of plastic strains (a true plastic zone that is 50% of the thickness of the cylinder wall) for the following locations in a pre-estimated plastic zone:

□ 12 elements in the plastic zone

Δ 14 elements in the plastic zone

◇ 16 elements in the plastic zone

Figure 15 shows how the objective value function changes when the number of mesh nodes increases. These results are from the 25% plastic zone in the cylinder that was subjected to only internal pressure. Testing began for 12 nodes and proceeded in a radial direction to 50 nodes. Figure 15 shows stabilization of the function when number of nodes is more than 35.

As noted previously, Figures 16 through 29 contain results for a combined load of cylinder. These figures indicate that a good convergence of results was obtained. Unfortunately, however, there is no analytical solution for a combined load of cylinder.

Figures 16 through 22 show plastic strains, residual stresses, and residual strains obtained for the following mesh nodes (p=0.46, M=57.0):

□ A 16-node mesh

Δ A 25-node mesh

◇ A 33-node mesh

Figures 23 through 29 show plastic strains, residual stresses, and residual strains obtained for the following mesh nodes (p=0.5, M=55.0):

□ A 25-node mesh

Δ A 33-node mesh

Suggestions for Further Tests

Thus far, only one-dimensional problems have been solved. It is very important to write programs for two-dimensional problems and to repeat some of the tests performed for one-dimensional problems to compare results. The main problem will be to obtain plastic and residual strains for railroad rails.

TABLE 1

true plastic zone that is 50% of cylinder wall thickness

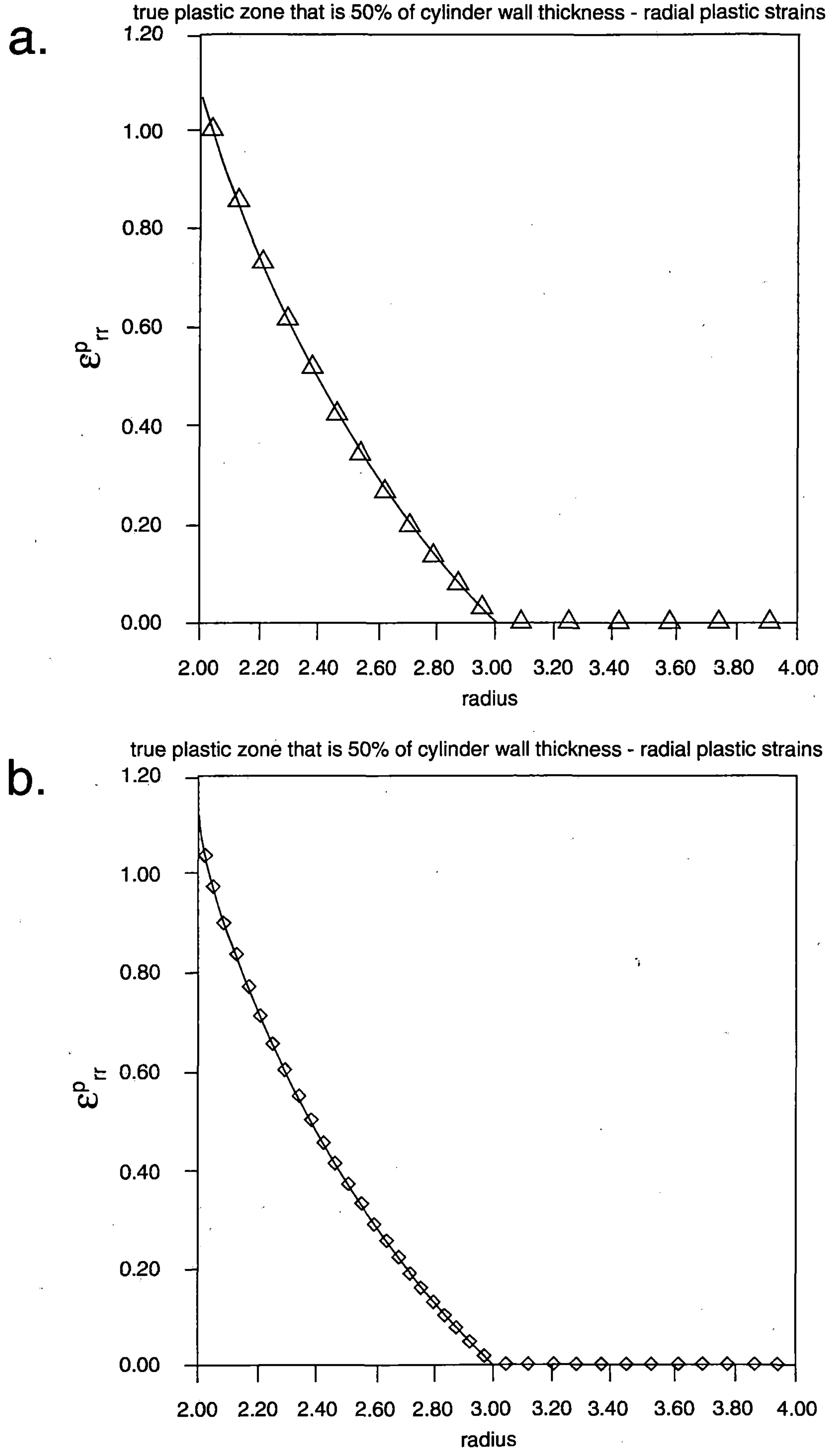
number of nodes removed from the true elastic zone and assigned to the pre-estimated plastic zone

NUMBER OF NODES	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12
12	ANAL	ANAL	SAT		KIN. ADM.							
16	ANAL		ANAL	SAT	SAT		KIN. ADM.					
20		ANAL		ANAL		SAT	SAT/ KIN.ADM		KIN. ADM.			
24		ANAL		ANAL			SAT		SAT/ KIN.ADM		KIN. ADM.	

true plastic zone that is 25% of cylinder wall thickness

number of nodes removed from the true elastic zone and assigned to the pre-estimated plastic zone

NUMBER OF NODES	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17
12	ANAL		ANAL	ANAL		ANAL		KIN. ADM.									
16		ANAL		ANAL		ANAL		ANAL	SAT		KIN. ADM.						
20		ANAL			ANAL			ANAL		SAT		SAT		KIN. ADM.			
24	ANAL					ANAL			ANAL			SAT			SAT/ KIN.ADM		KIN. ADM.



a.
true plastic zone that is 50% of cylinder wall thickness - radial plastic strains
1.20
1.00
0.80
0.60
0.40
0.20
0.00
ε^p_{rr}
2.00 2.20 2.40 2.60 2.80 3.00 3.20 3.40 3.60 3.80 4.00
radius
b.
true plastic zone that is 50% of cylinder wall thickness - radial plastic strains
1.20
1.00
0.80
0.60
0.40
0.20
0.00
ε^p_{rr}
2.00 2.20 2.40 2.60 2.80 3.00 3.20 3.40 3.60 3.80 4.00
radius

Figure 1.

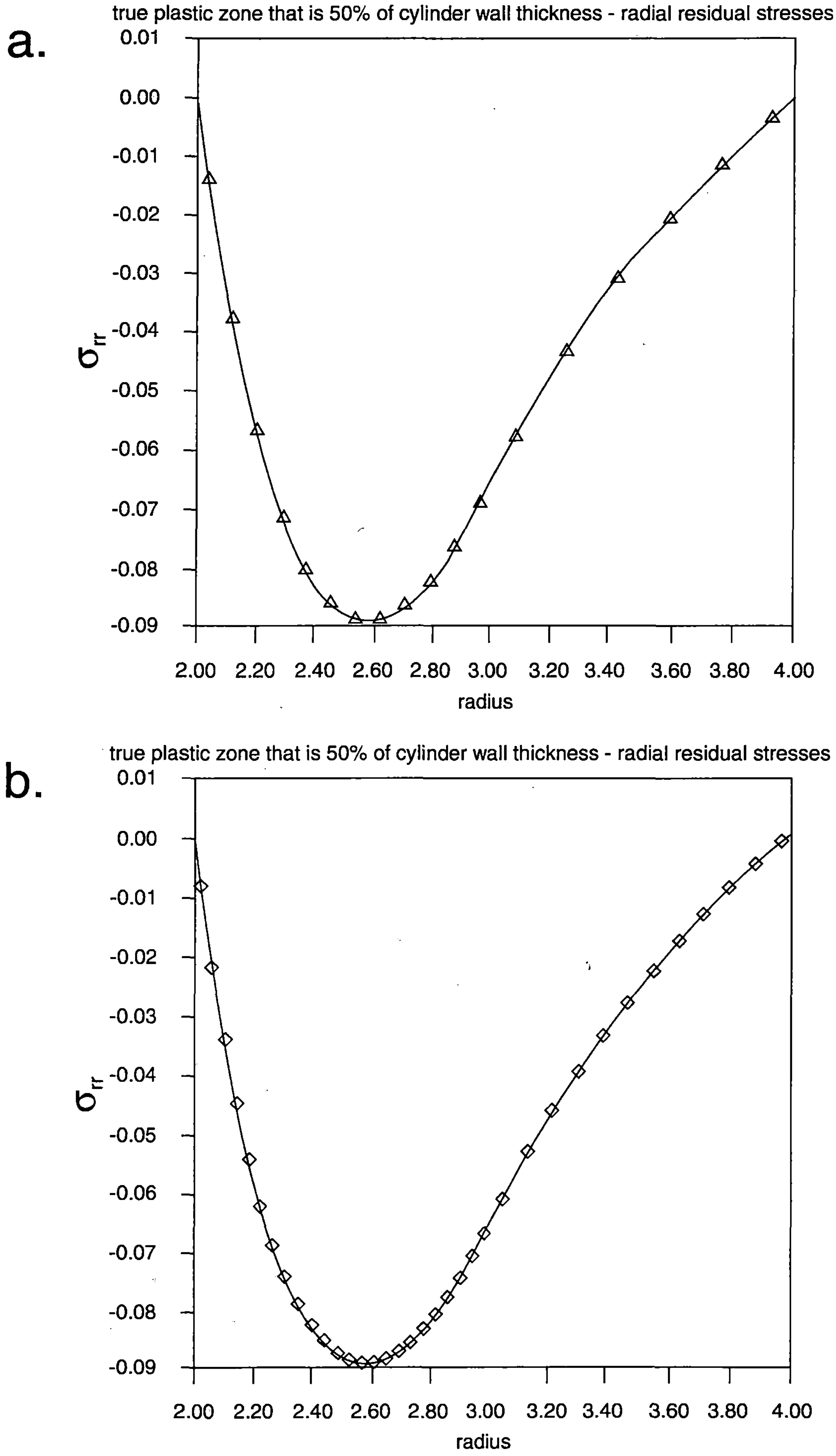
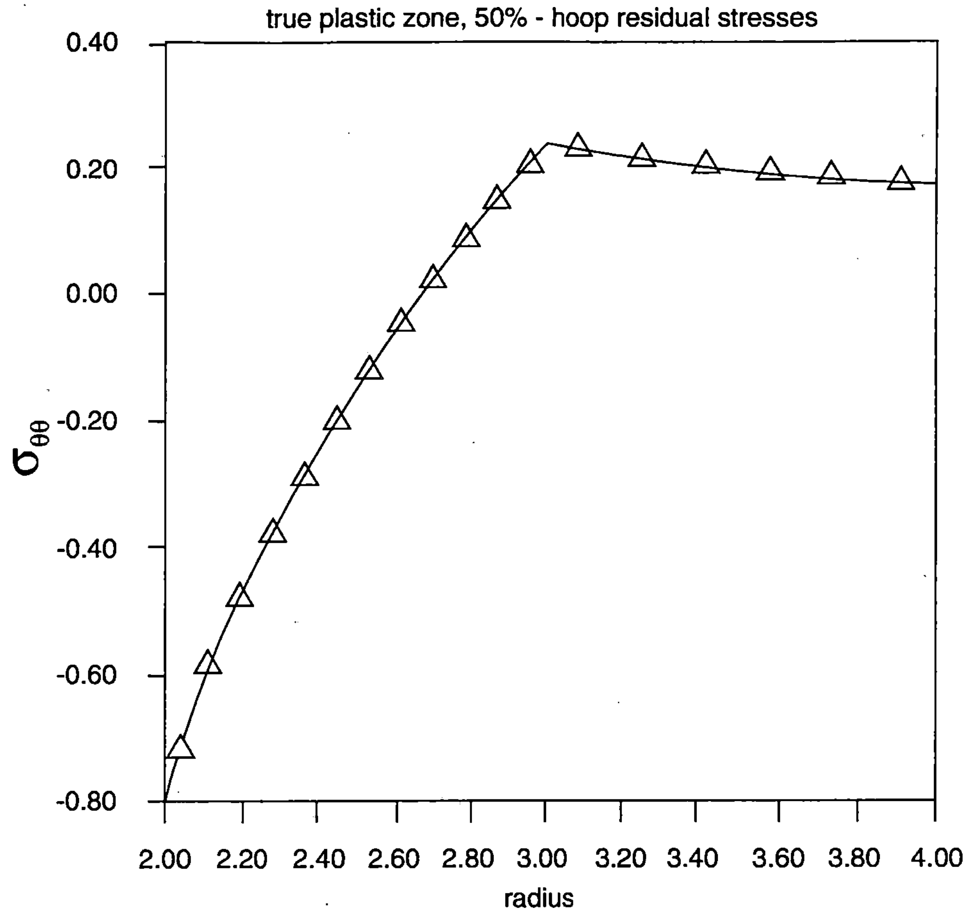

a.
true plastic zone that is 50% of cylinder wall thickness - radial residual stresses
0.01
0.00
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
-0.09
σ_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius
b.
true plastic zone that is 50% of cylinder wall thickness - radial residual stresses
0.01
0.00
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
-0.09
σ_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

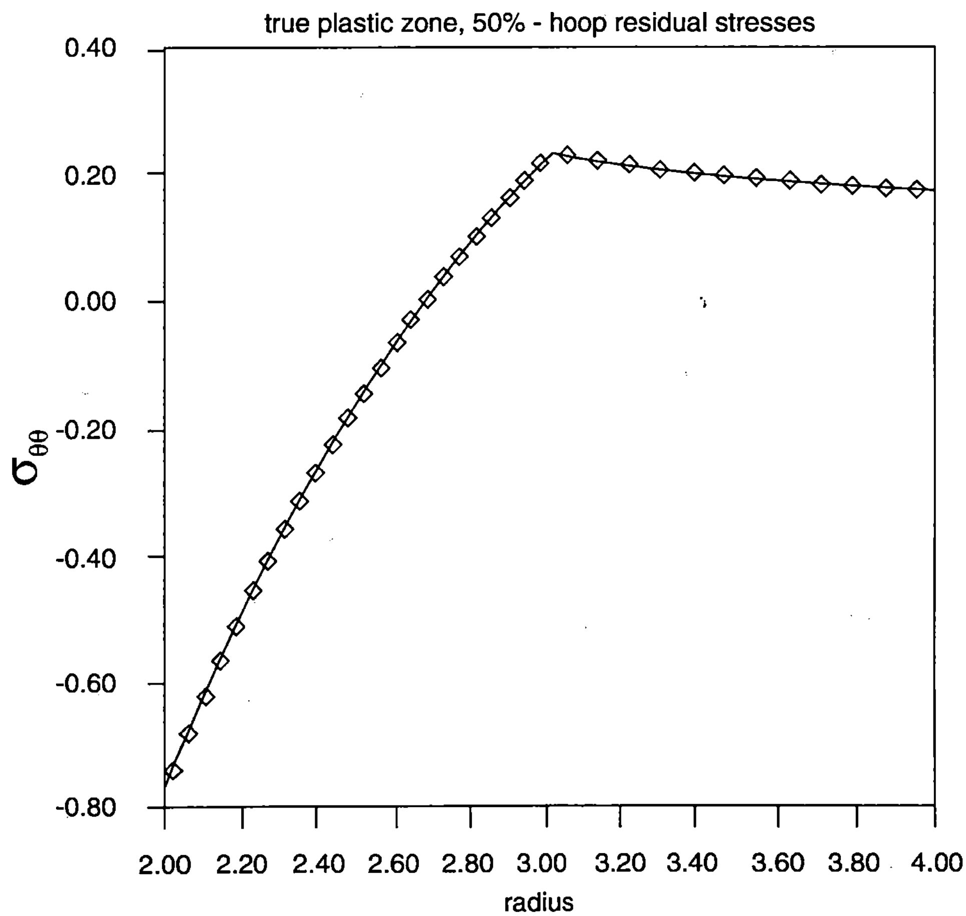
Figure 2.

a.



true plastic zone, 50% - hoop residual stresses
0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
$\sigma_{\theta\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

b.



true plastic zone, 50% - hoop residual stresses
0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
$\sigma_{\theta\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 3.

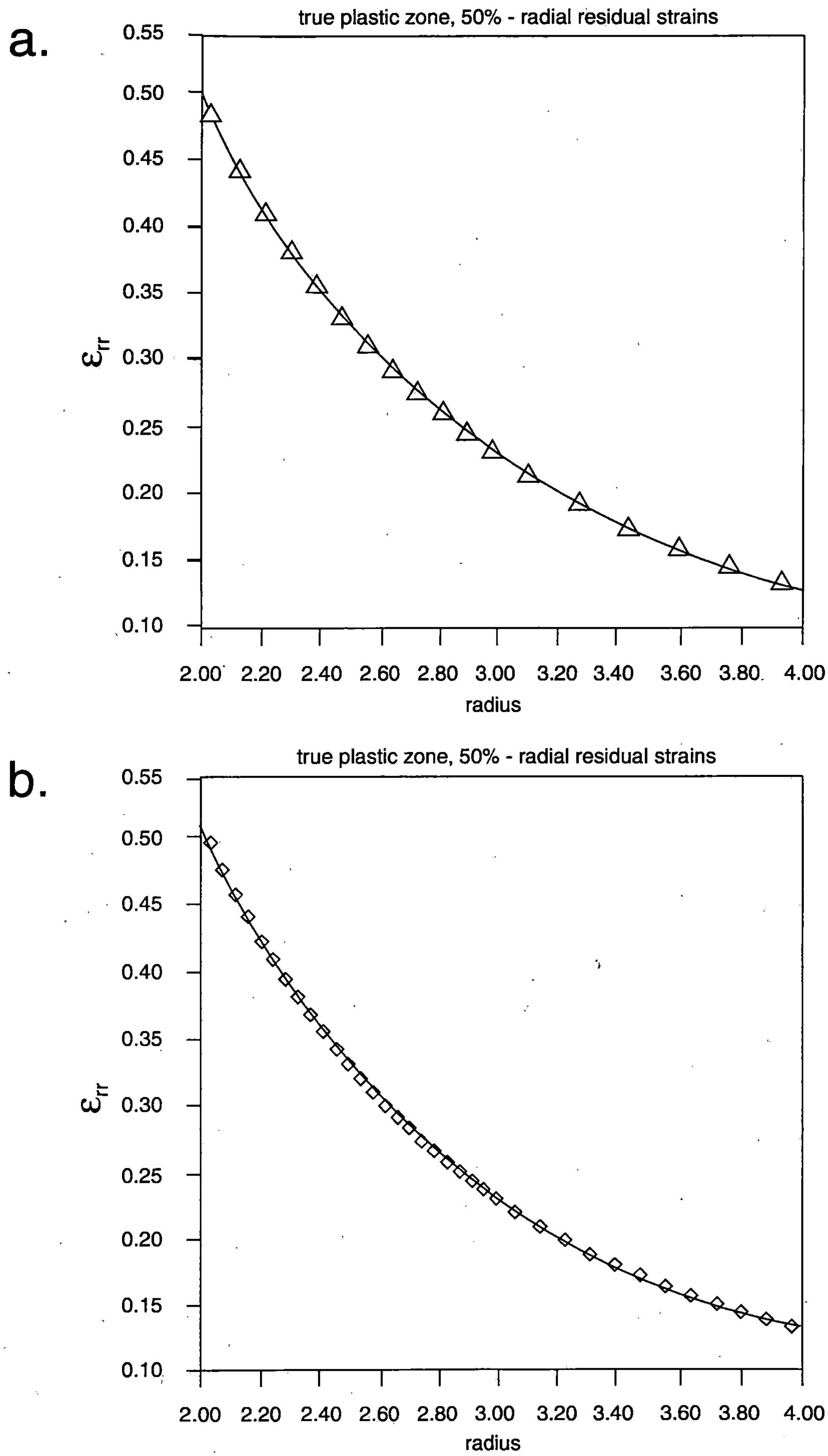

a.
true plastic zone, 50% - radial residual strains
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
ε_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius
b.
true plastic zone, 50% - radial residual strains
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
ε_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 4.

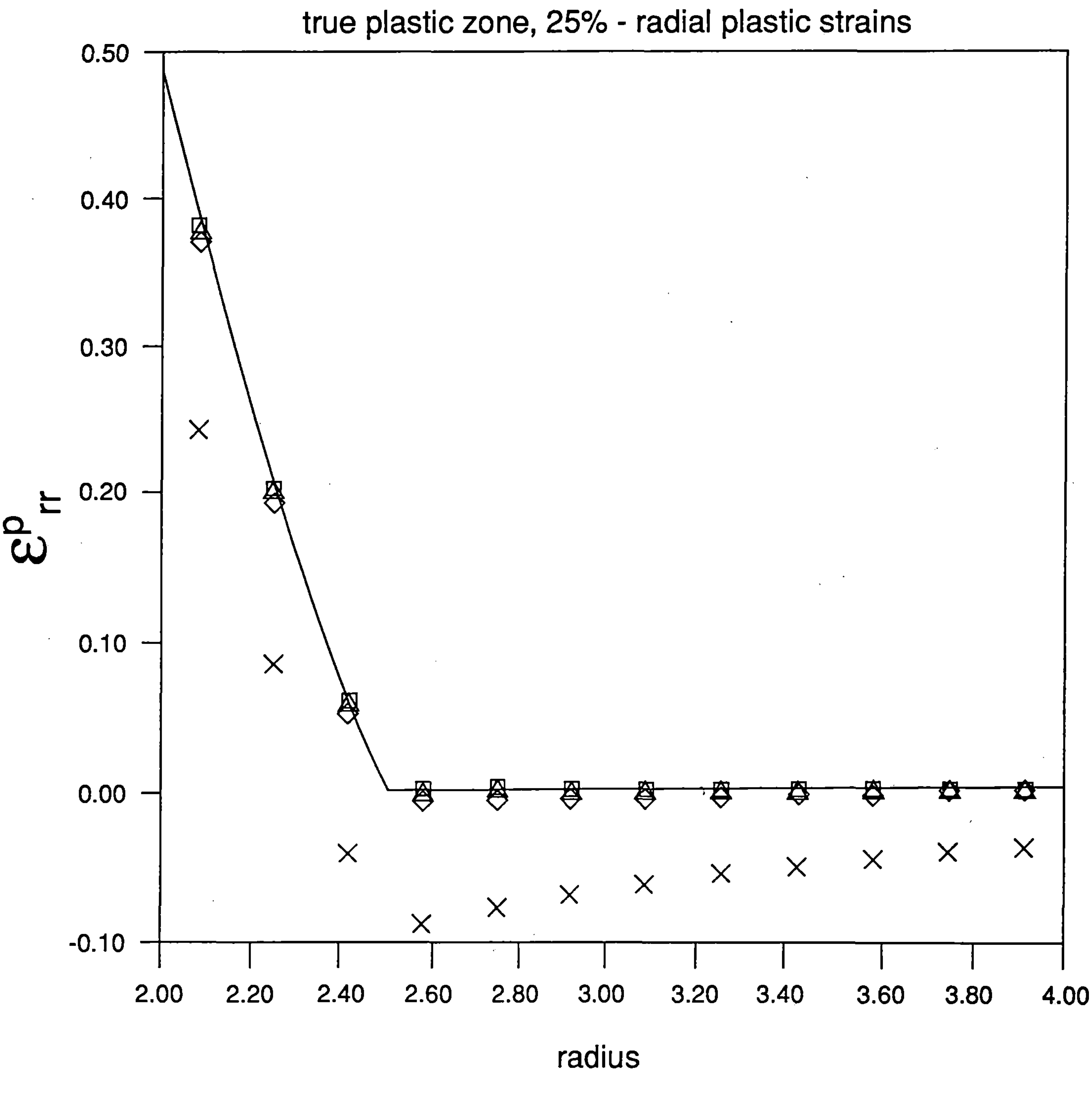
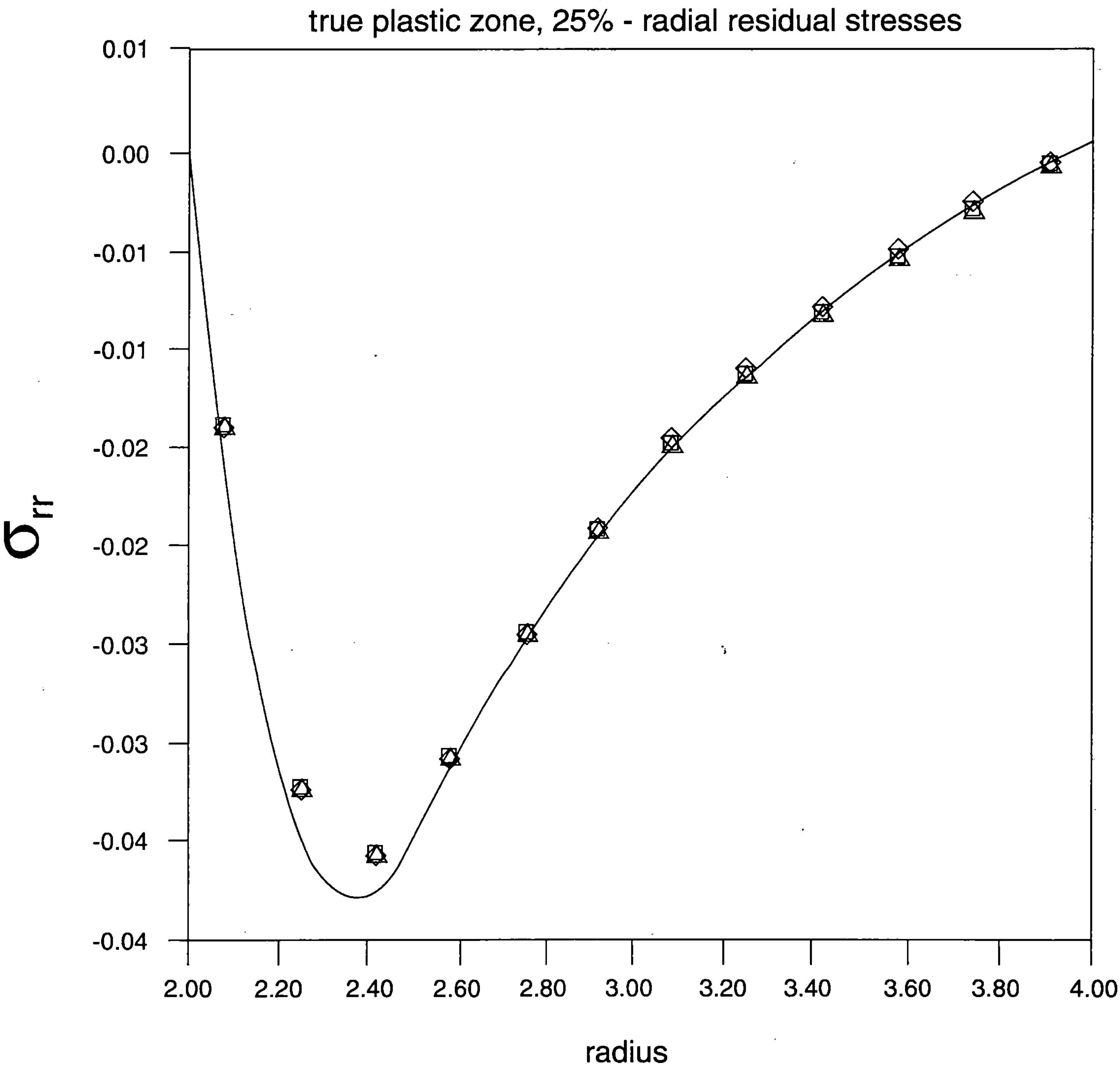

true plastic zone, 25% - radial plastic strains
0.50
0.40
0.30
0.20
0.10
0.00
-0.10
ε^p_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 5.



true plastic zone, 25% - radial residual stresses
0.01
0.00
-0.01
-0.01
-0.02
-0.02
-0.03
-0.03
-0.04
-0.04
σ_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 6.

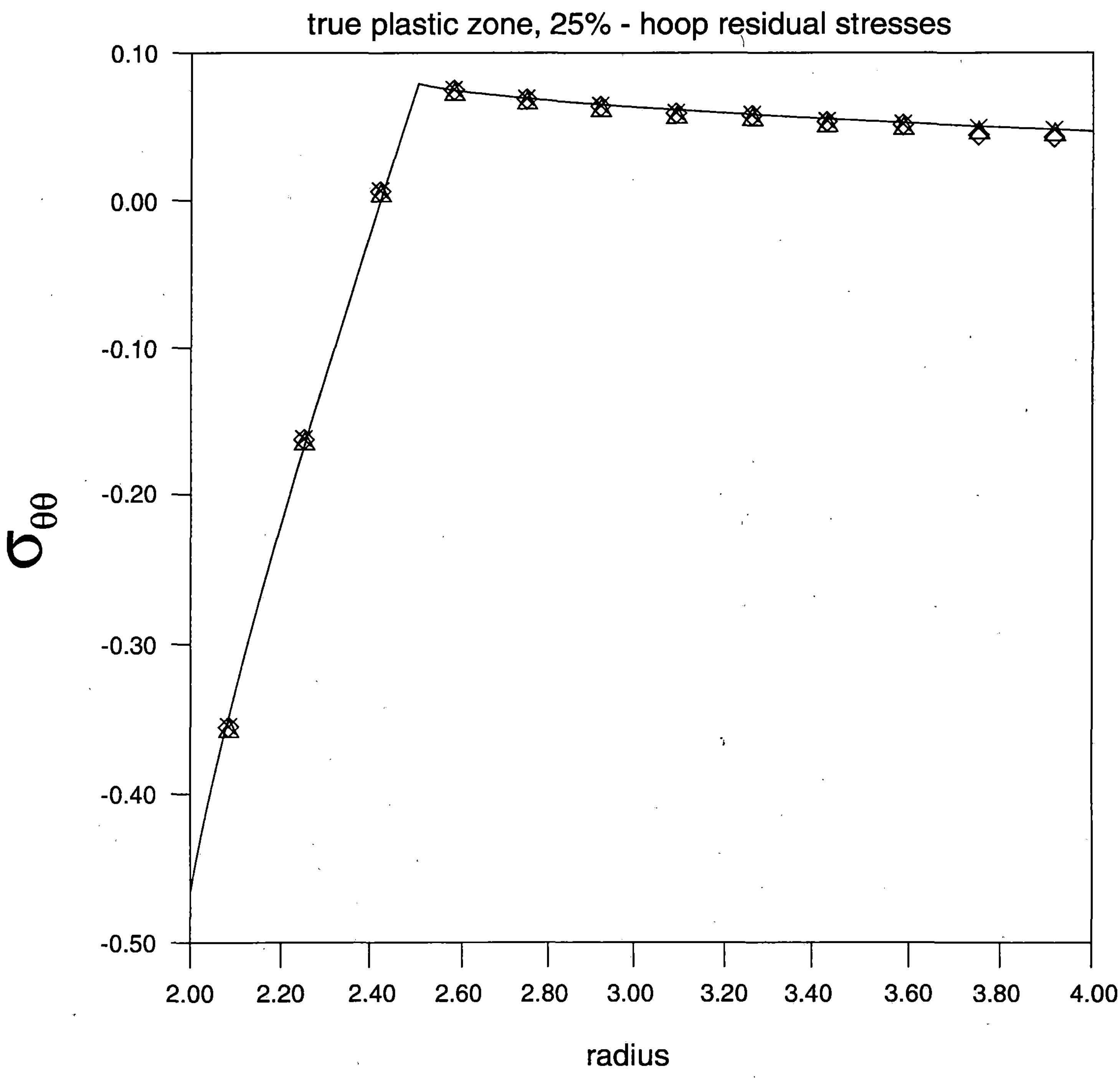
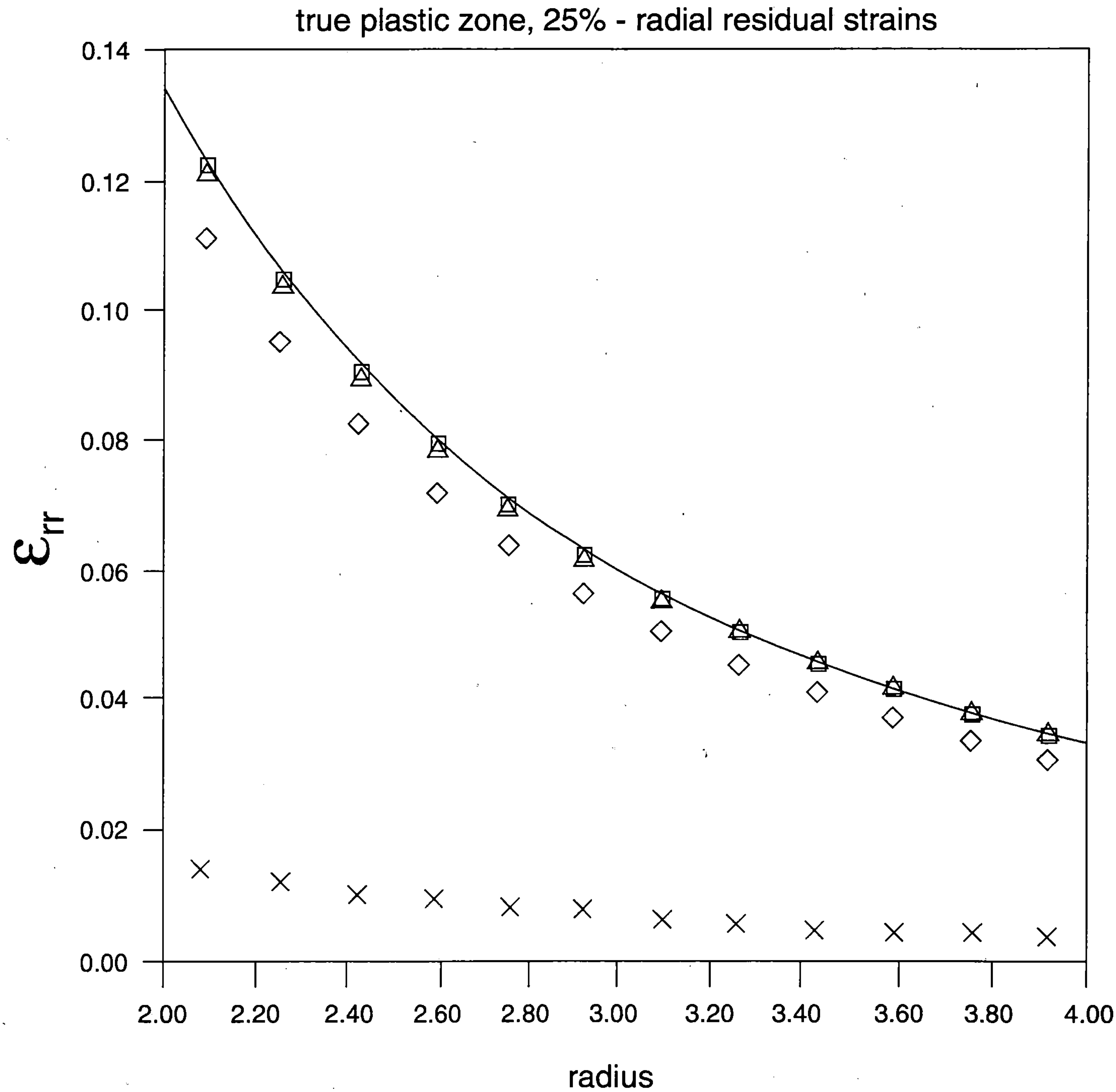

true plastic zone, 25% - hoop residual stresses
0.10
0.00
-0.10
-0.20
-0.30
-0.40
-0.50
$\sigma_{\theta\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 7.



true plastic zone, 25% - radial residual strains
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
ε_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 8.

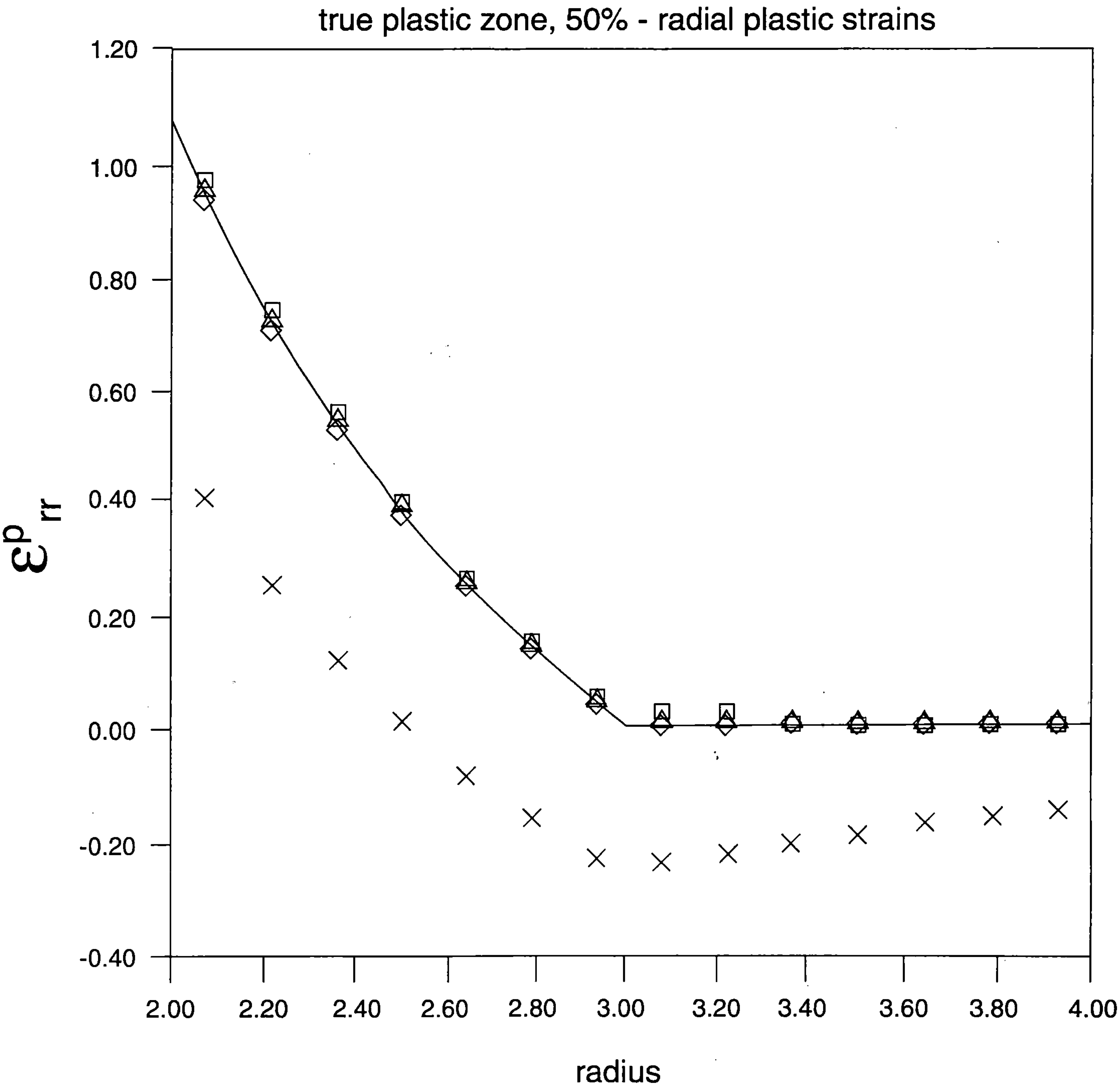

true plastic zone, 50% - radial plastic strains
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
ε^p_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 9.

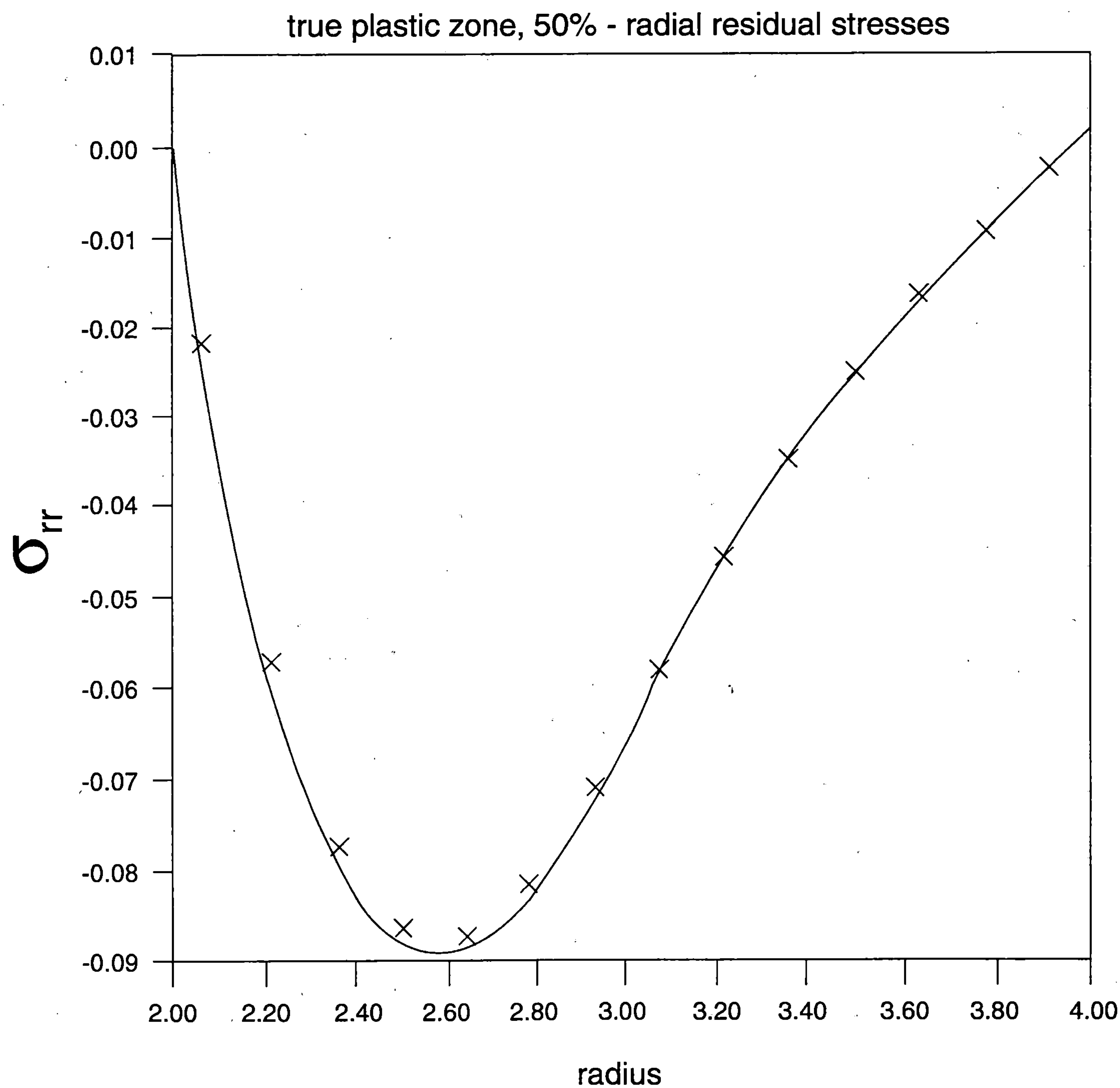
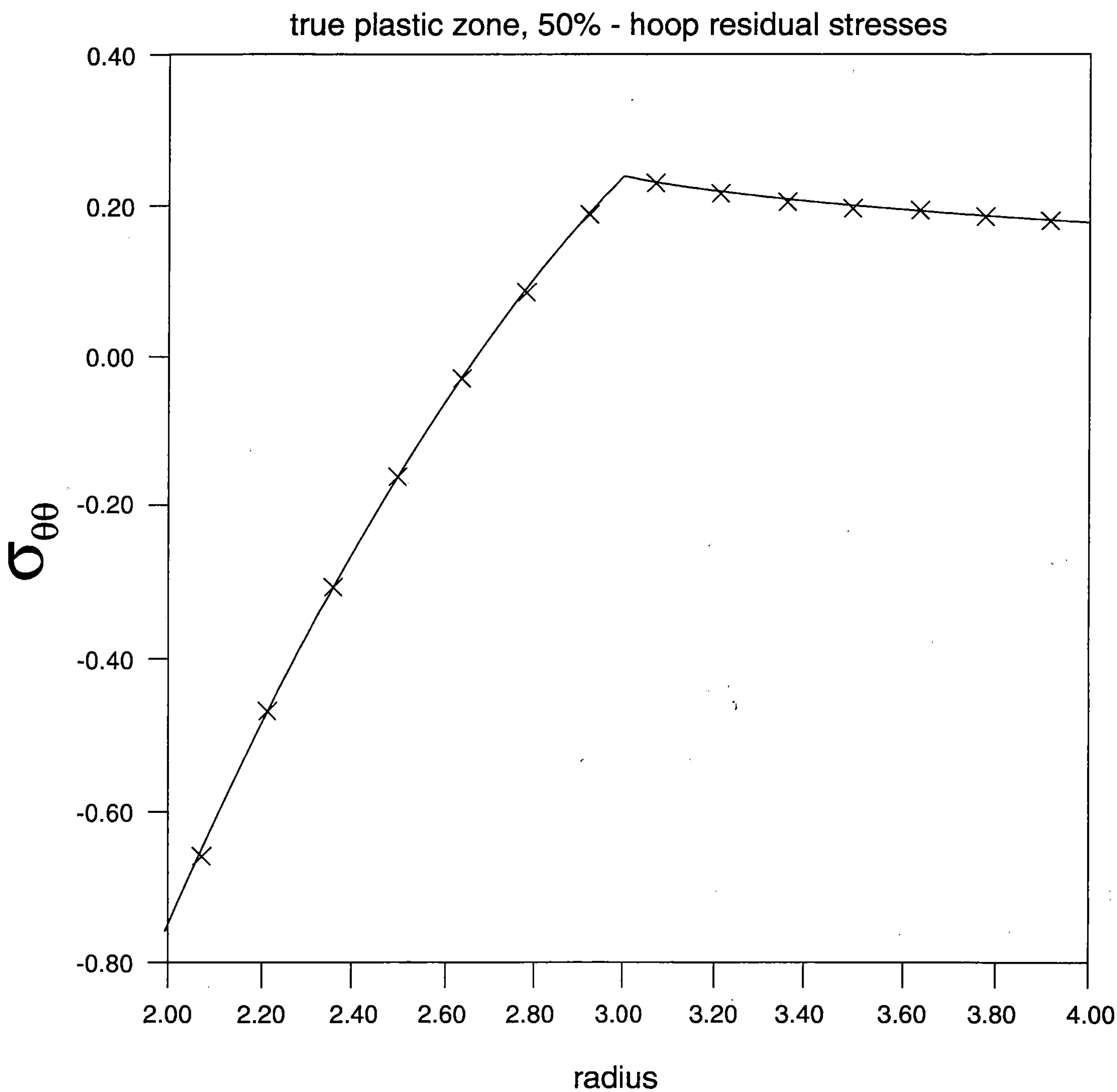
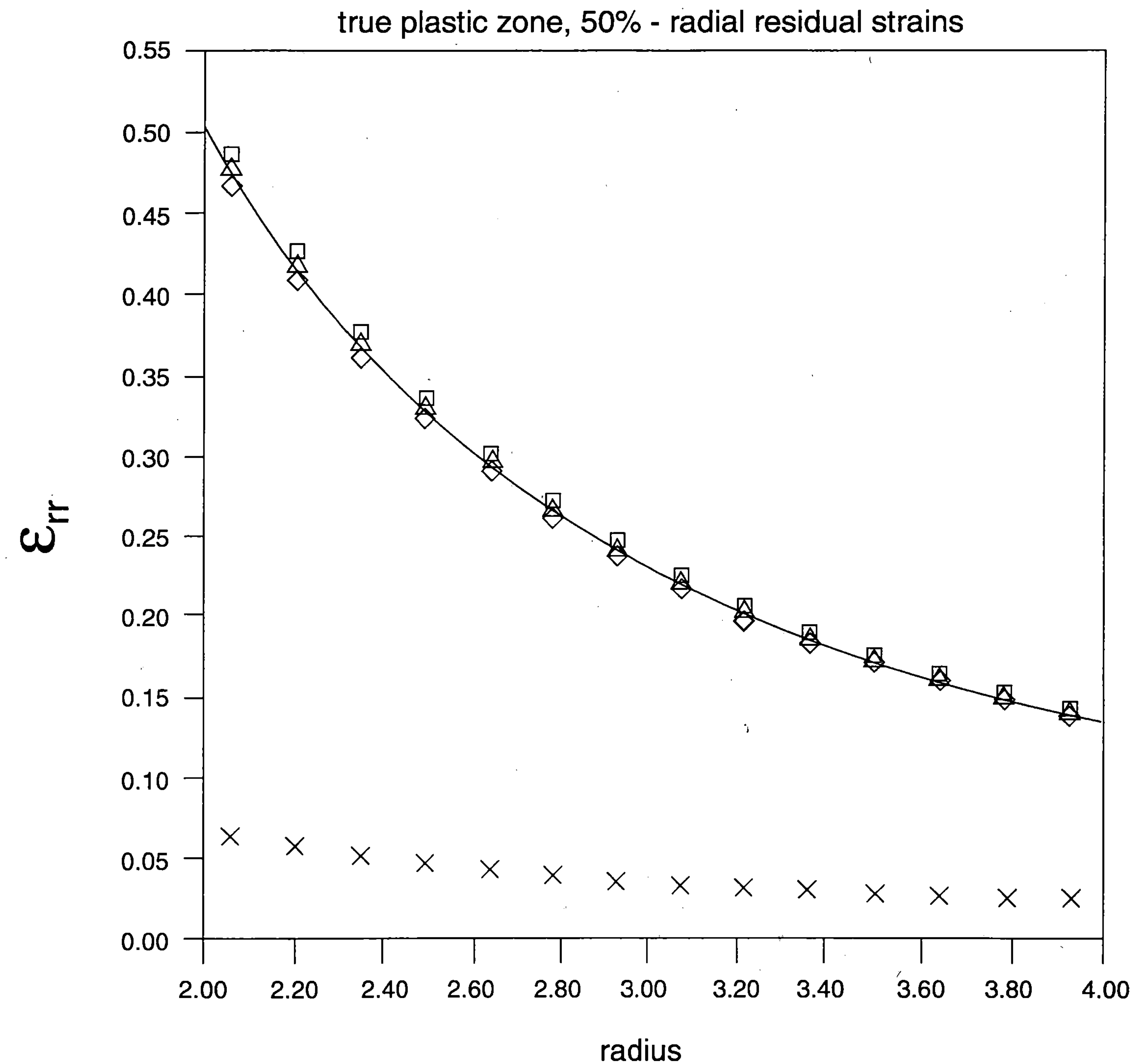

true plastic zone, 50% - radial residual stresses
0.01
0.00
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
-0.09
σ_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 10.



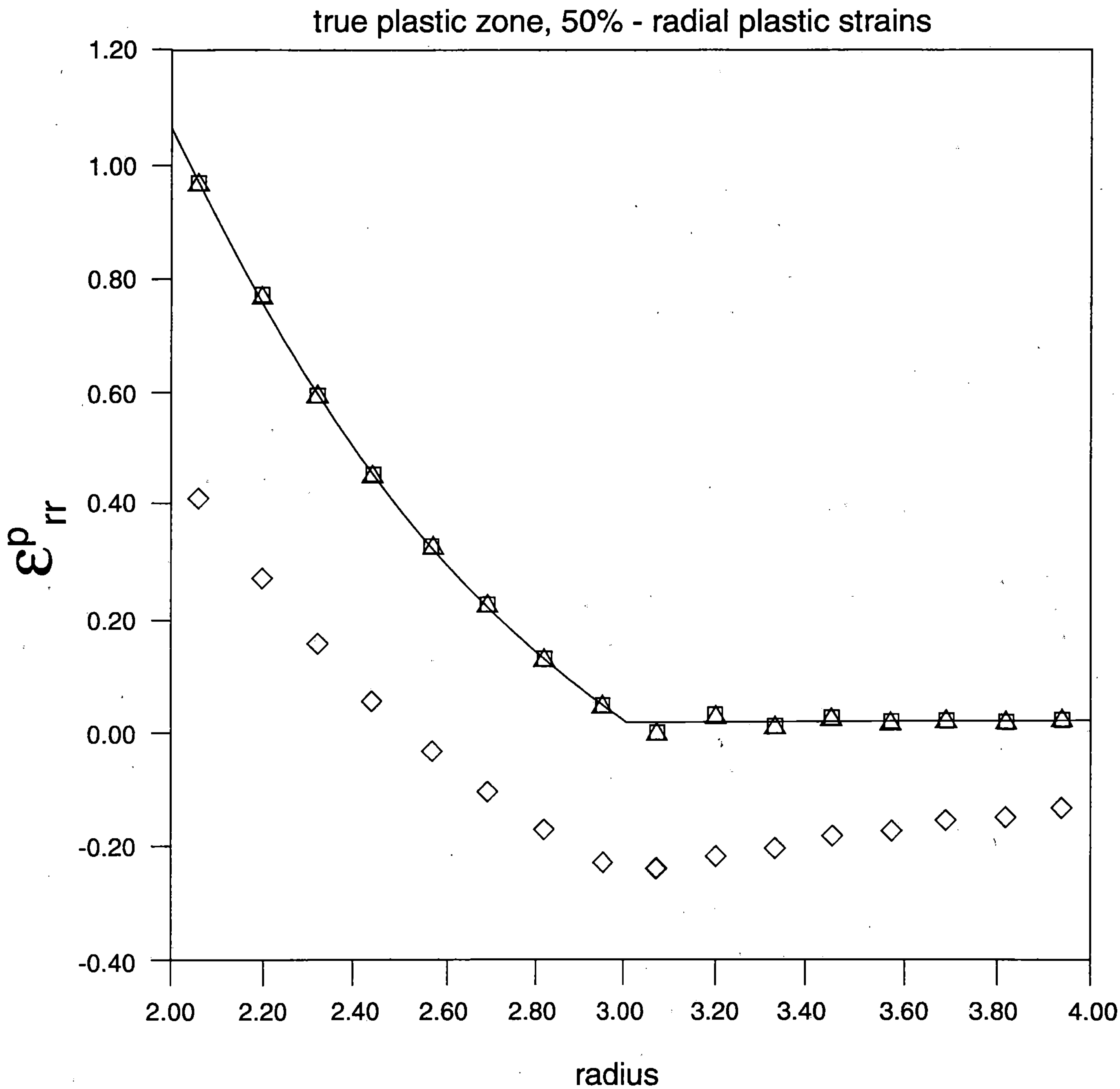
true plastic zone, 50% - hoop residual stresses
0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
$\sigma_{\theta\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 11.



true plastic zone, 50% - radial residual strains
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
ε_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 12.



true plastic zone, 50% - radial plastic strains
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
ε^p_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 13.

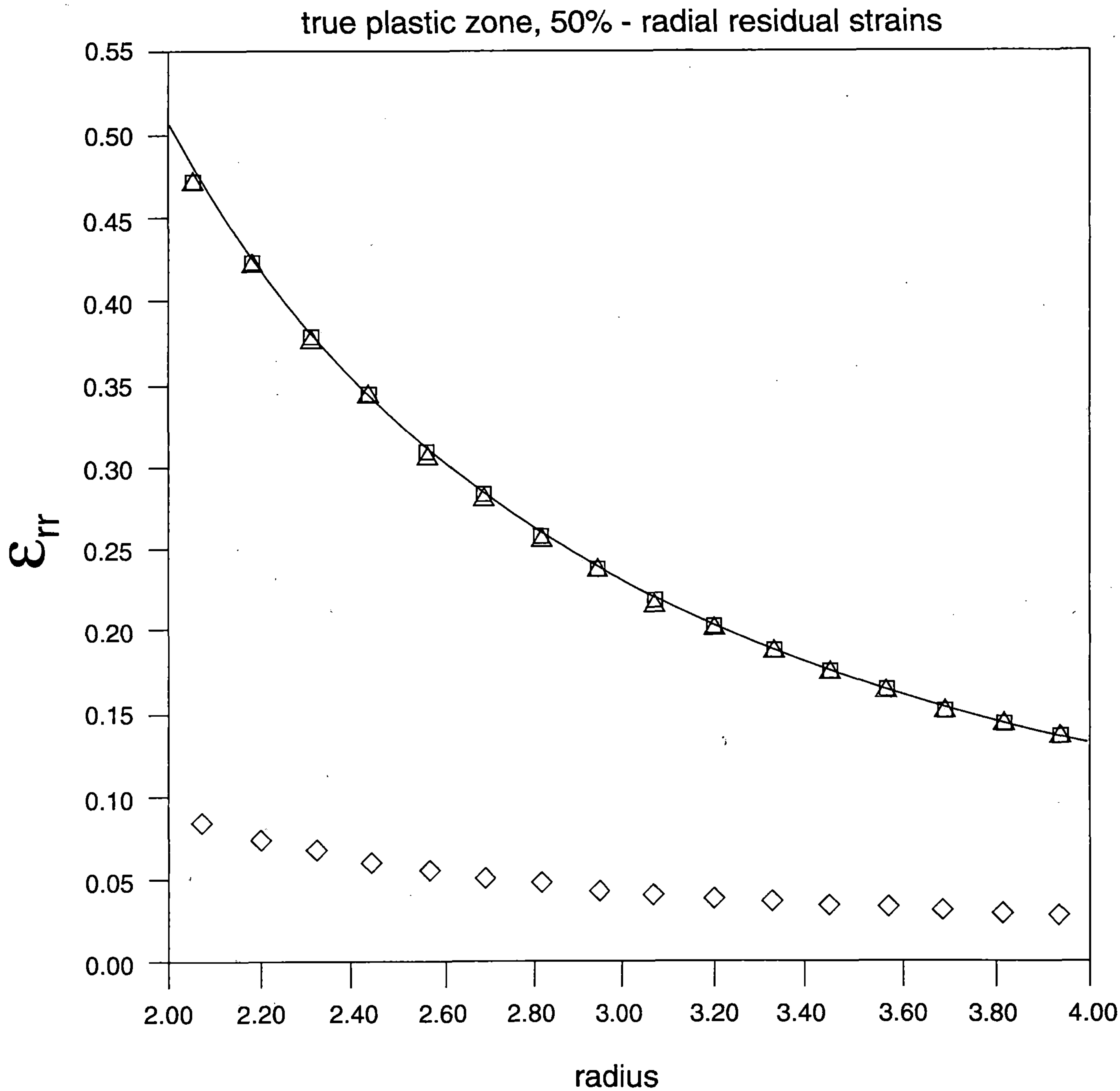

true plastic zone, 50% - radial residual strains
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
ε_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 14.

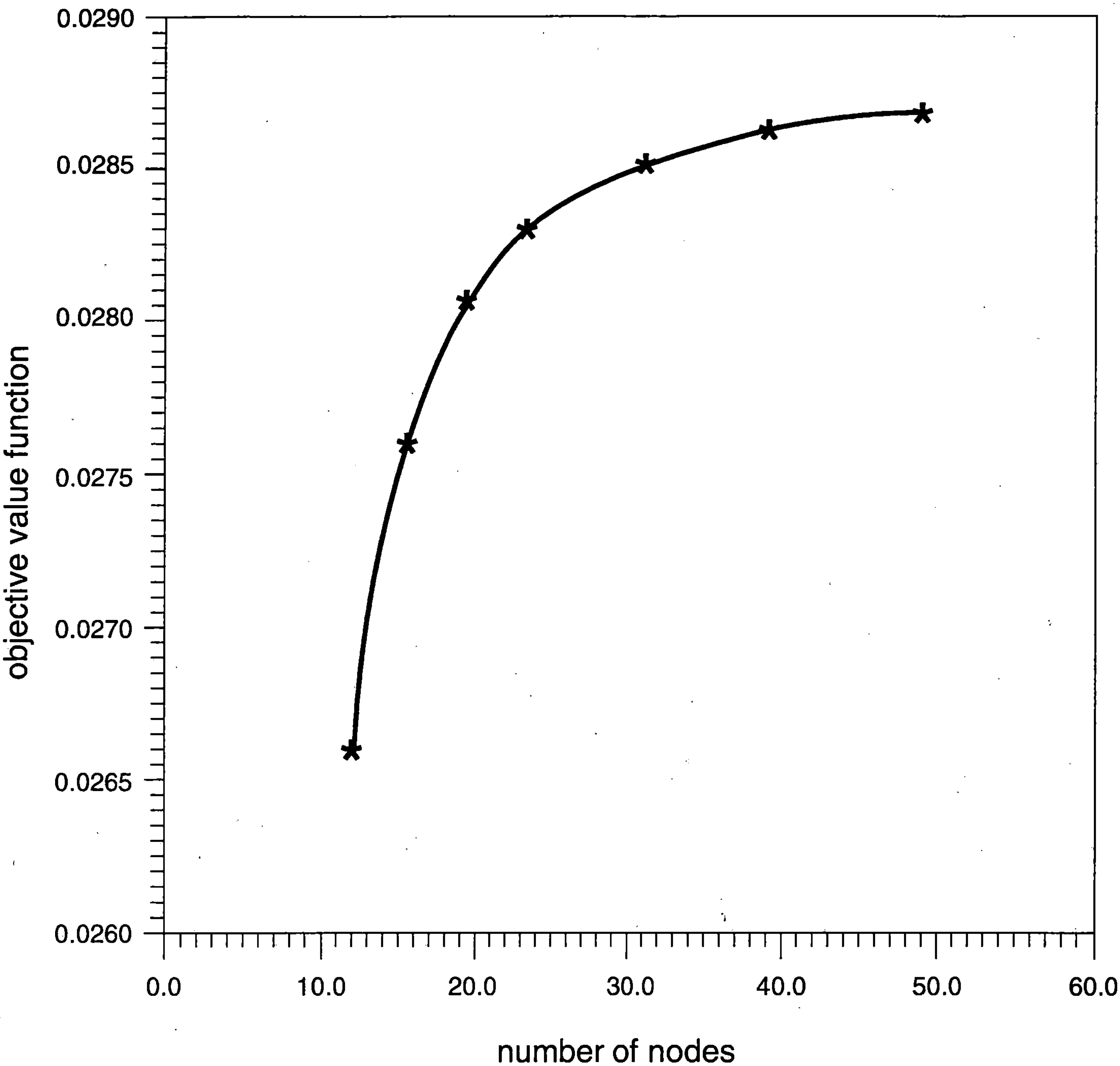
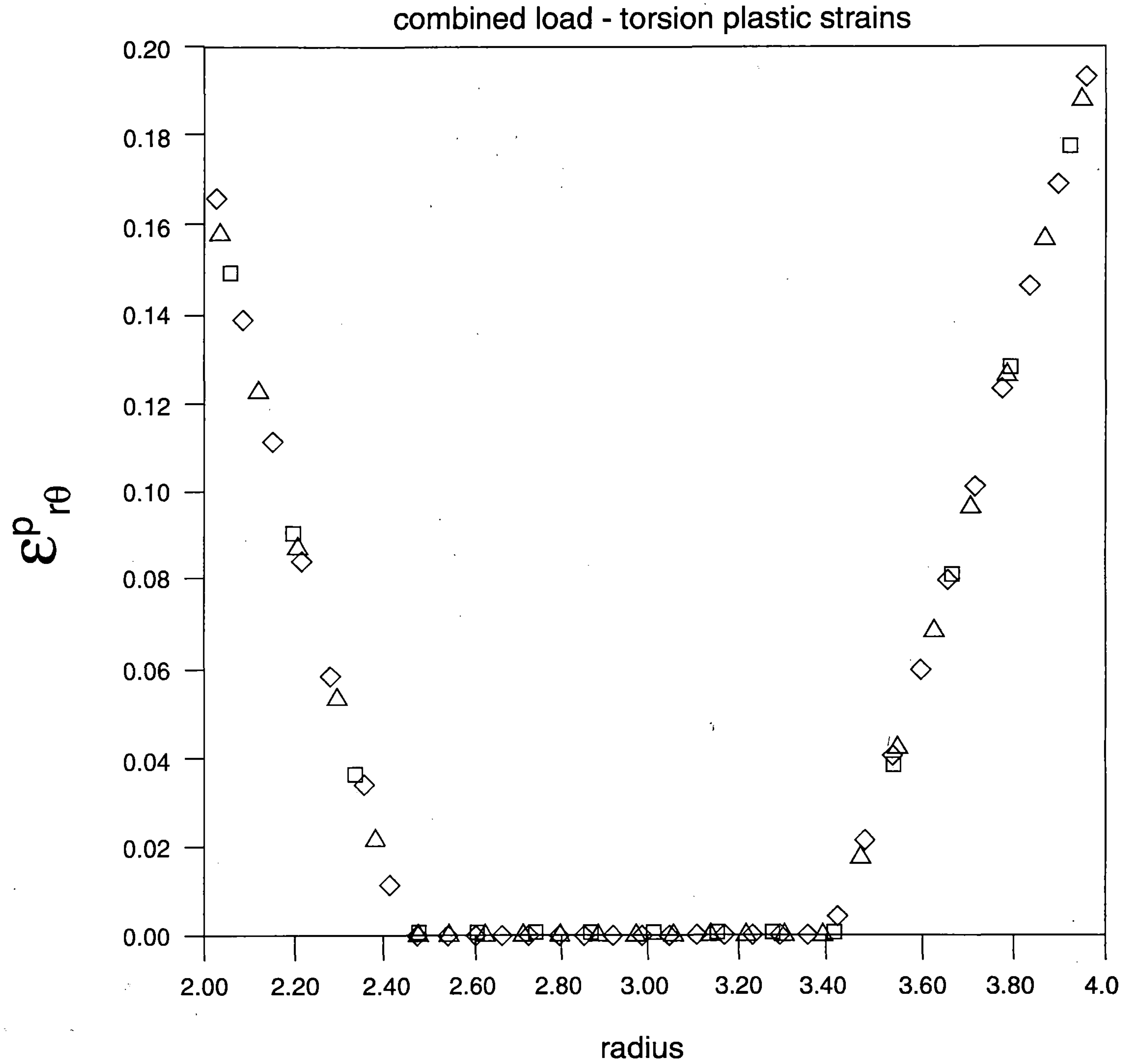

0.0290
0.0285
0.0280
0.0275
0.0270
0.0265
0.0260
objective value function
0.0
10.0
20.0
30.0
40.0
50.0
60.0
number of nodes

Figure 15.



combined load - torsion plastic strains
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
$\varepsilon^p_{r\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.0
radius

Figure 16.

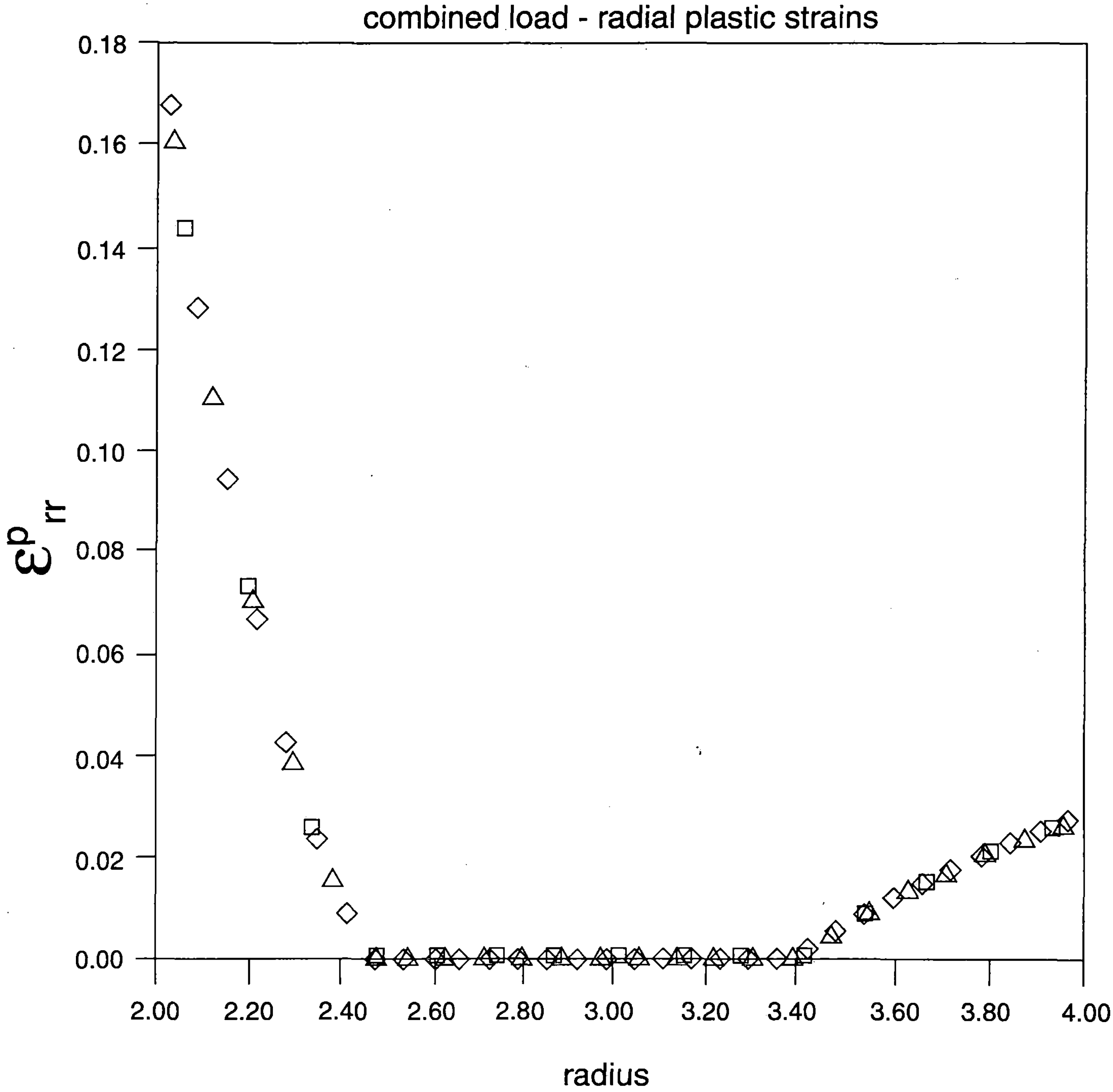
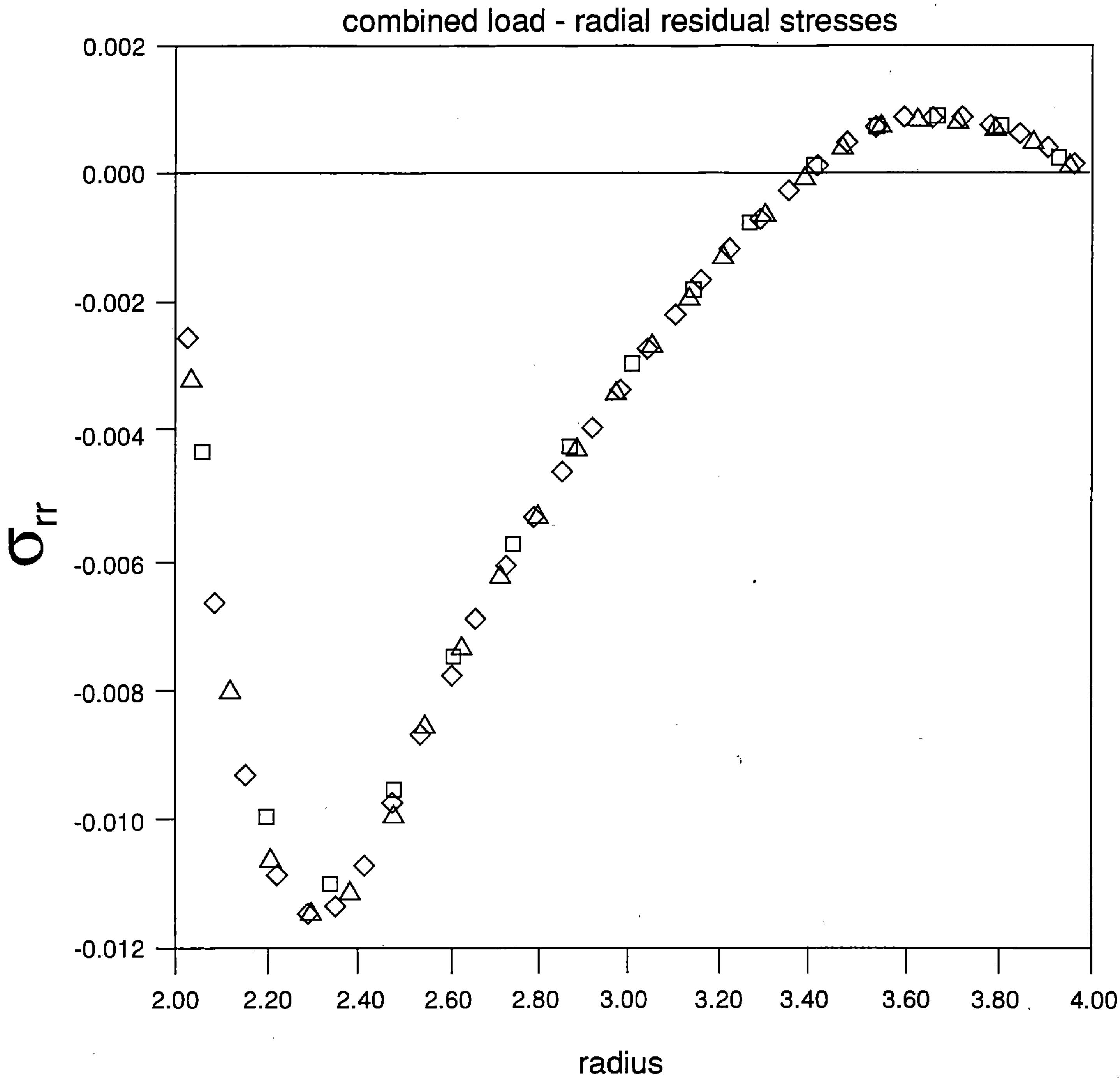

combined load - radial plastic strains
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
ε^{p}_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 17.



combined load - radial residual stresses
0.002
0.000
-0.002
-0.004
-0.006
-0.008
-0.010
-0.012
σ_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 18.

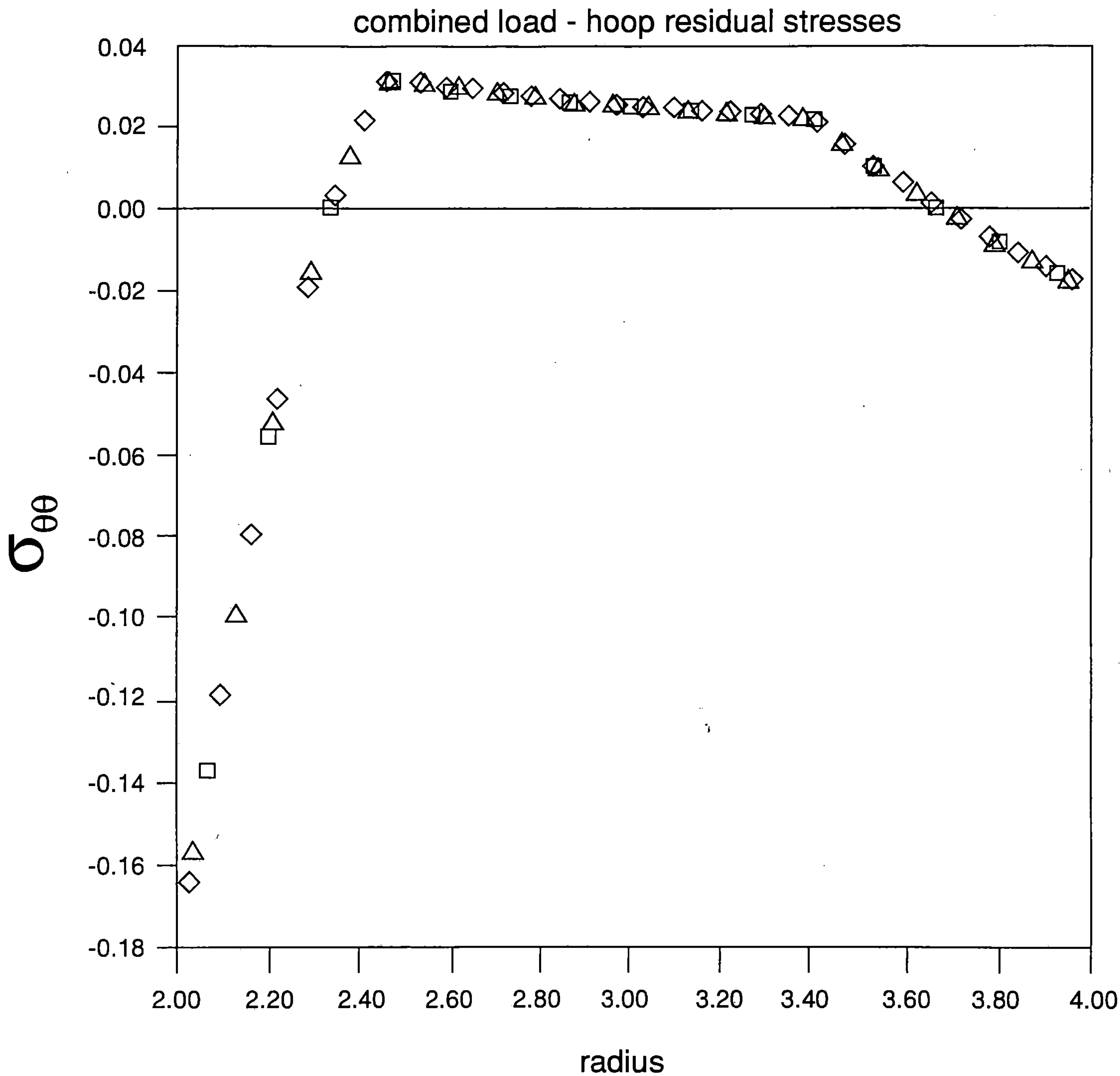

combined load - hoop residual stresses
0.04
0.02
0.00
-0.02
-0.04
-0.06
-0.08
-0.10
-0.12
-0.14
-0.16
-0.18
$\sigma_{\theta\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 19.

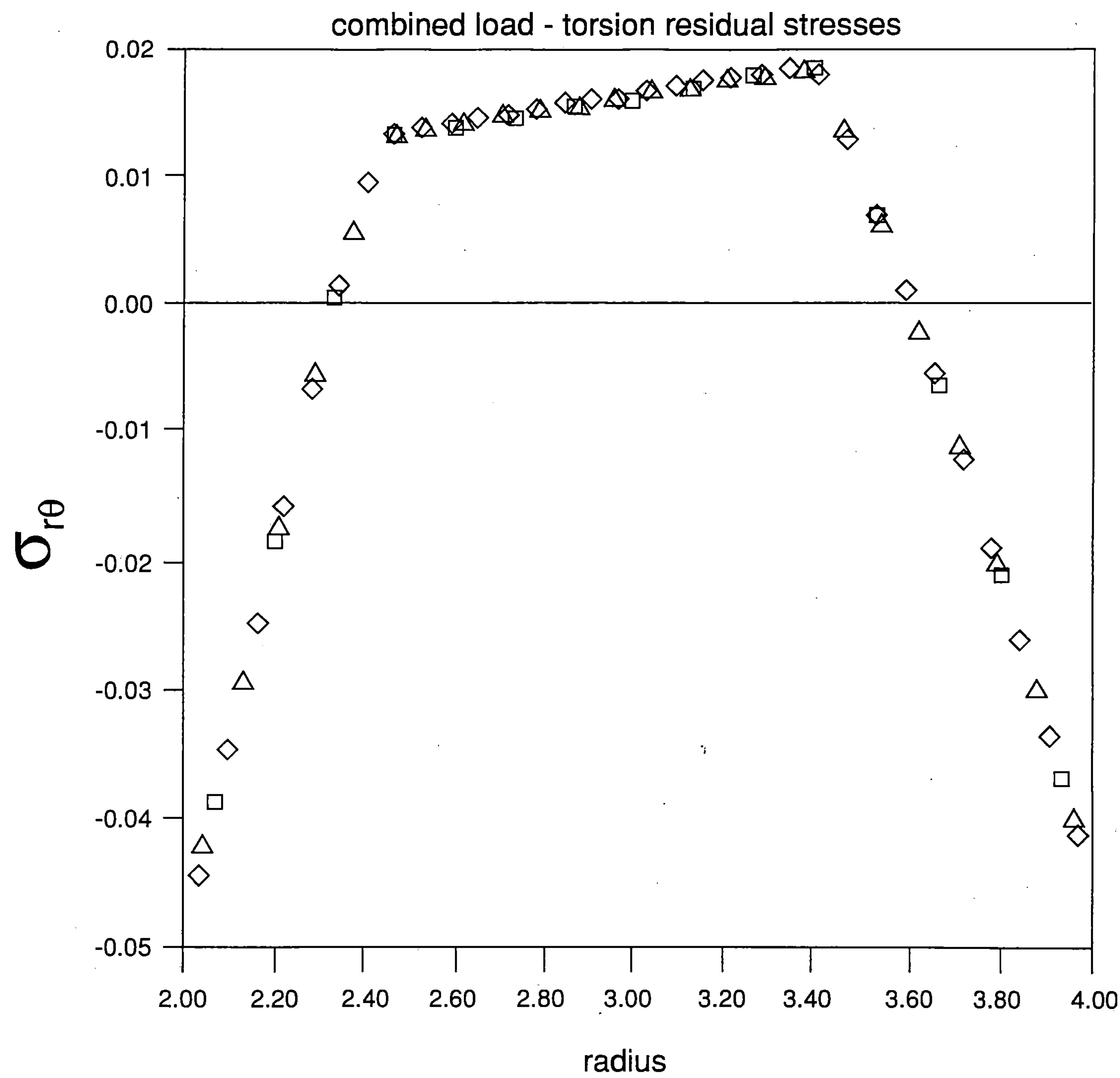
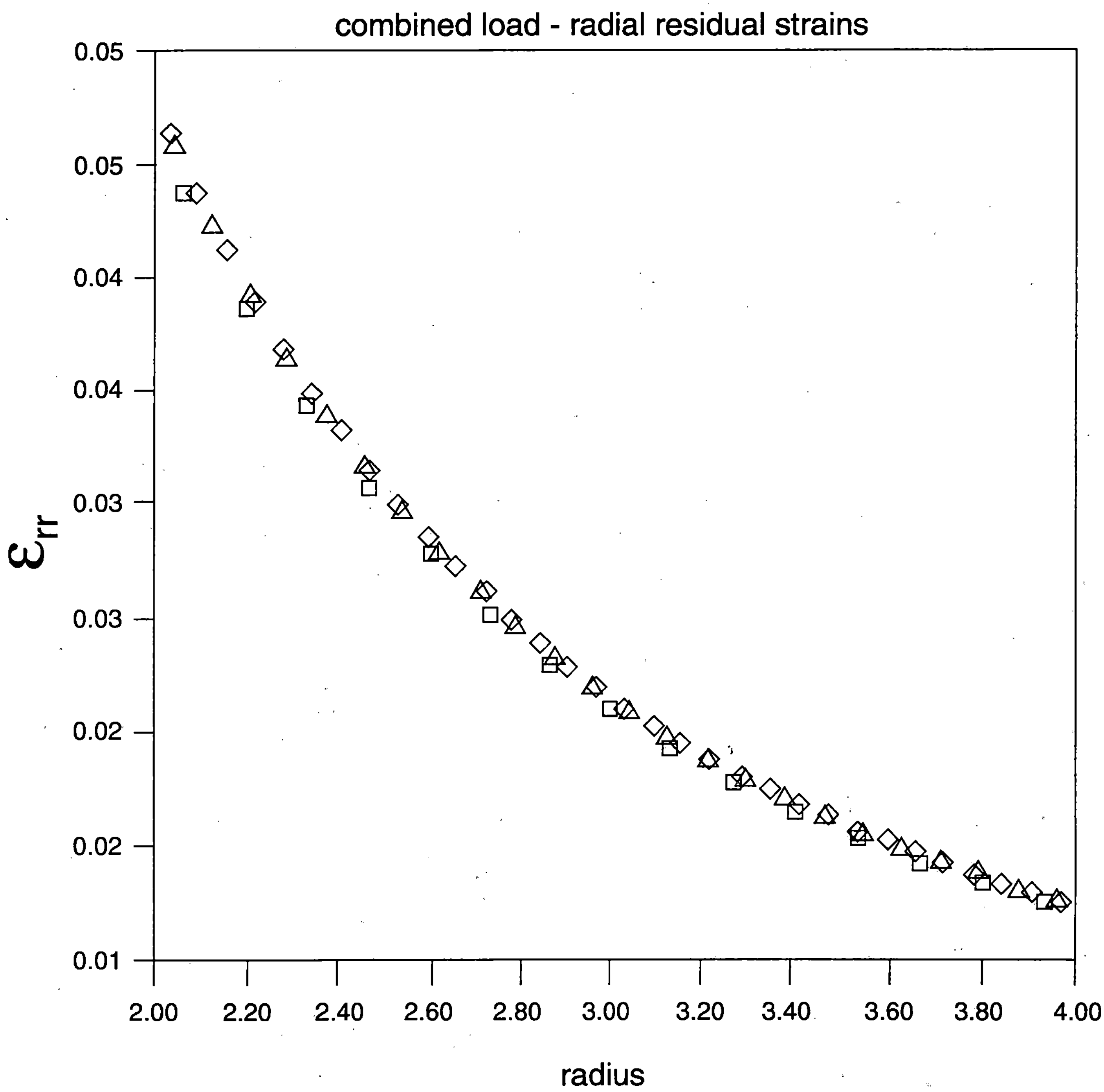
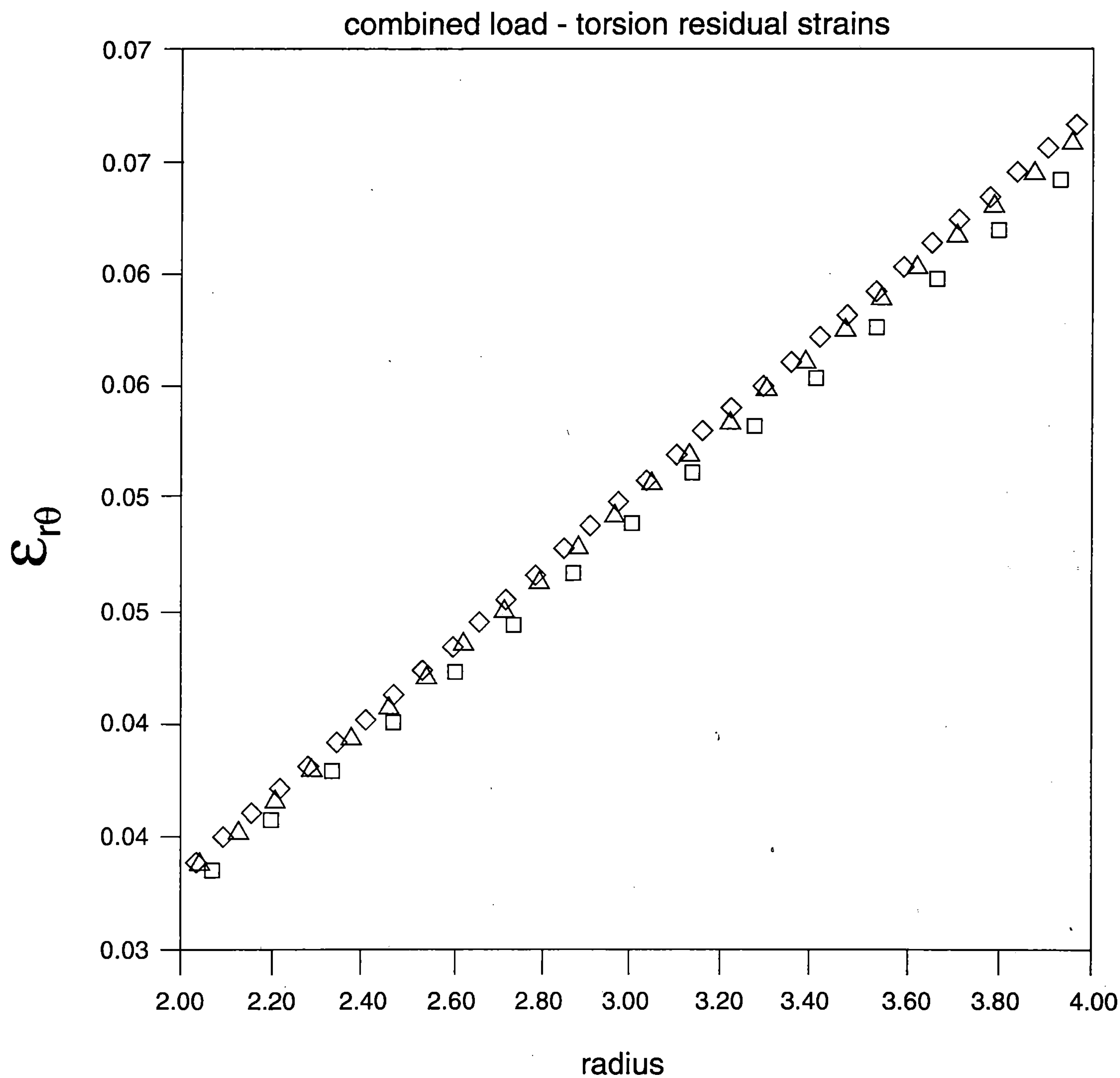

combined load - torsion residual stresses
0.02
0.01
0.00
-0.01
-0.02
-0.03
-0.04
-0.05
σrθ
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 20.



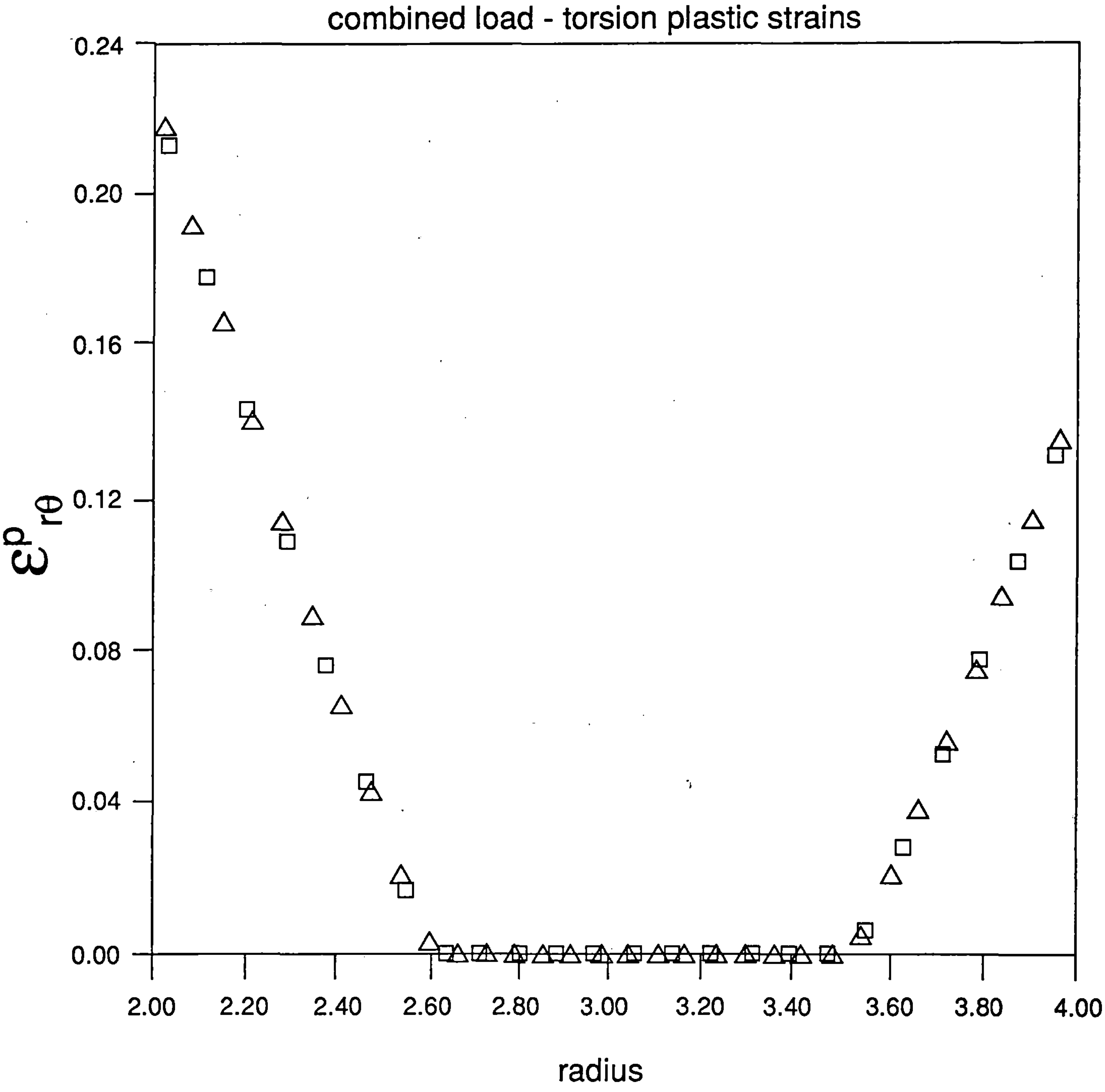
combined load - radial residual strains
0.05
0.05
0.04
0.04
0.03
0.03
0.02
0.02
0.01
ε_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 21.



combined load - torsion residual strains
0.07
0.07
0.06
0.06
0.05
0.05
0.04
0.04
0.03
$\varepsilon_{r\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 22.



combined load - torsion plastic strains
0.24
0.20
0.16
0.12
0.08
0.04
0.00
$\varepsilon^p_{r\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 23.

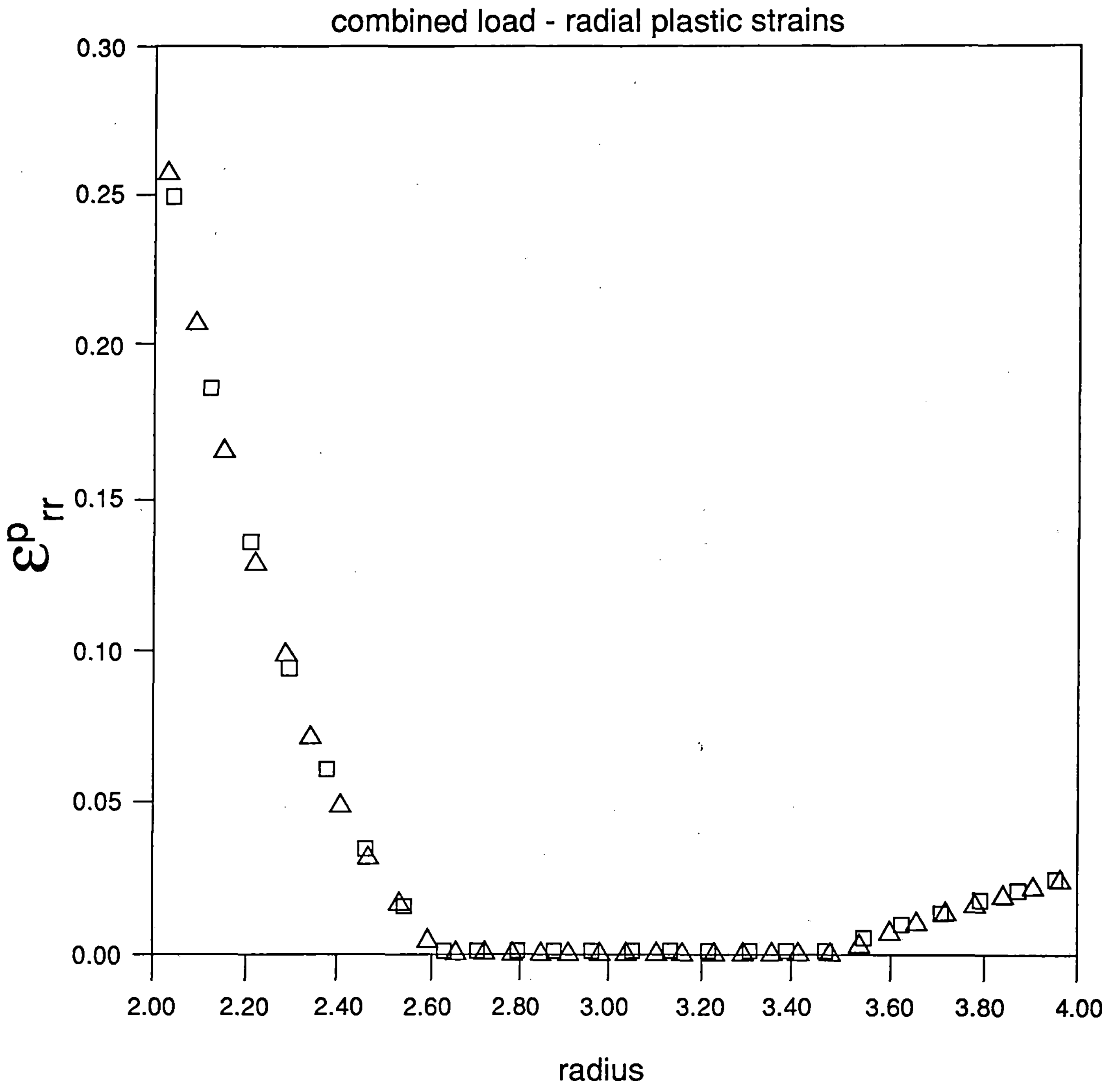

combined load - radial plastic strains
0.30
0.25
0.20
0.15
0.10
0.05
0.00
ε^p_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 24.

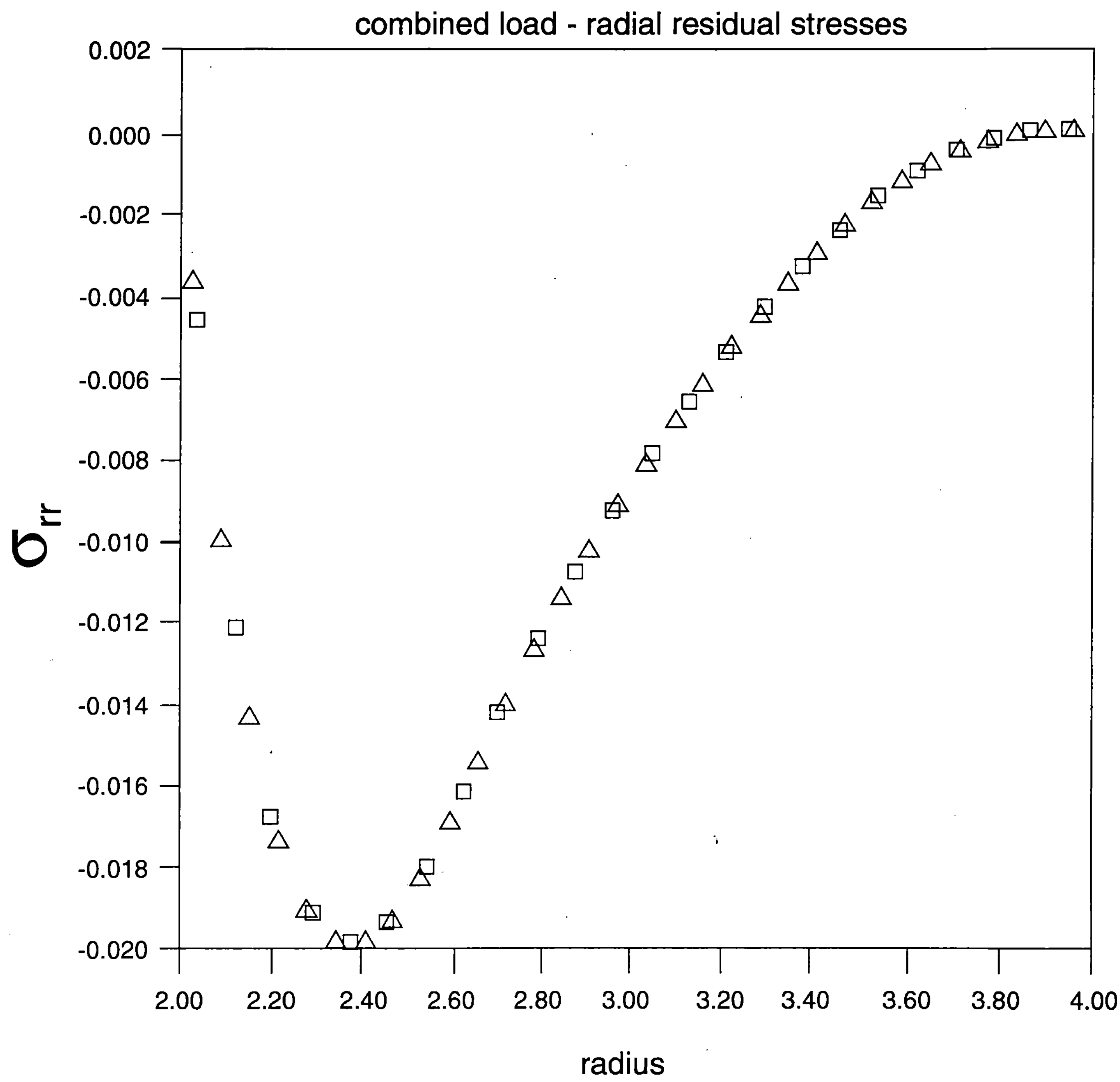

combined load - radial residual stresses
0.002
0.000
-0.002
-0.004
-0.006
-0.008
-0.010
-0.012
-0.014
-0.016
-0.018
-0.020
σrr
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 25.

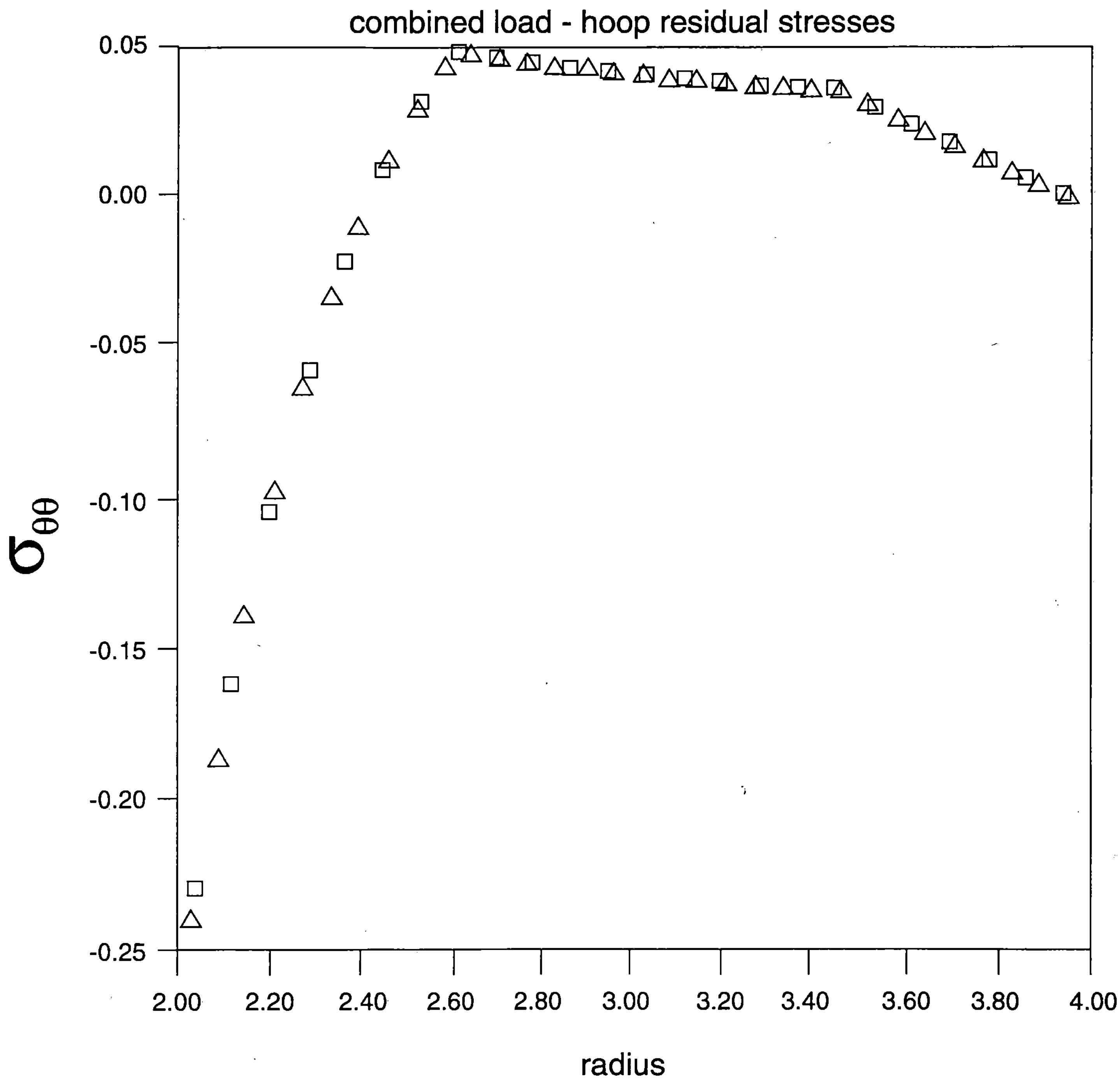
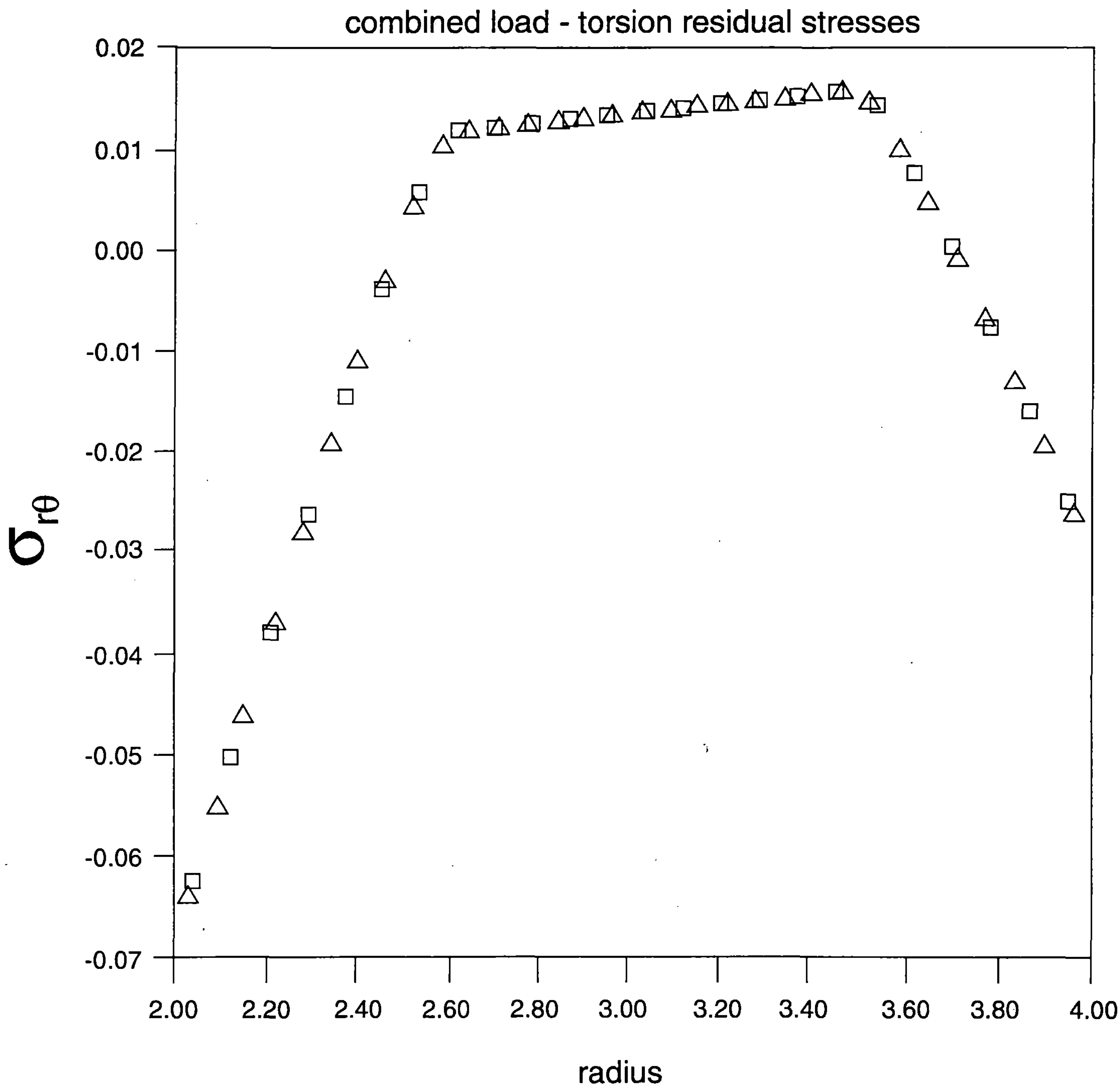

combined load - hoop residual stresses
0.05
0.00
-0.05
-0.10
-0.15
-0.20
-0.25
$\sigma_{\theta\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 26.



combined load - torsion residual stresses
0.02
0.01
0.00
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
$\sigma_{r\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 27.

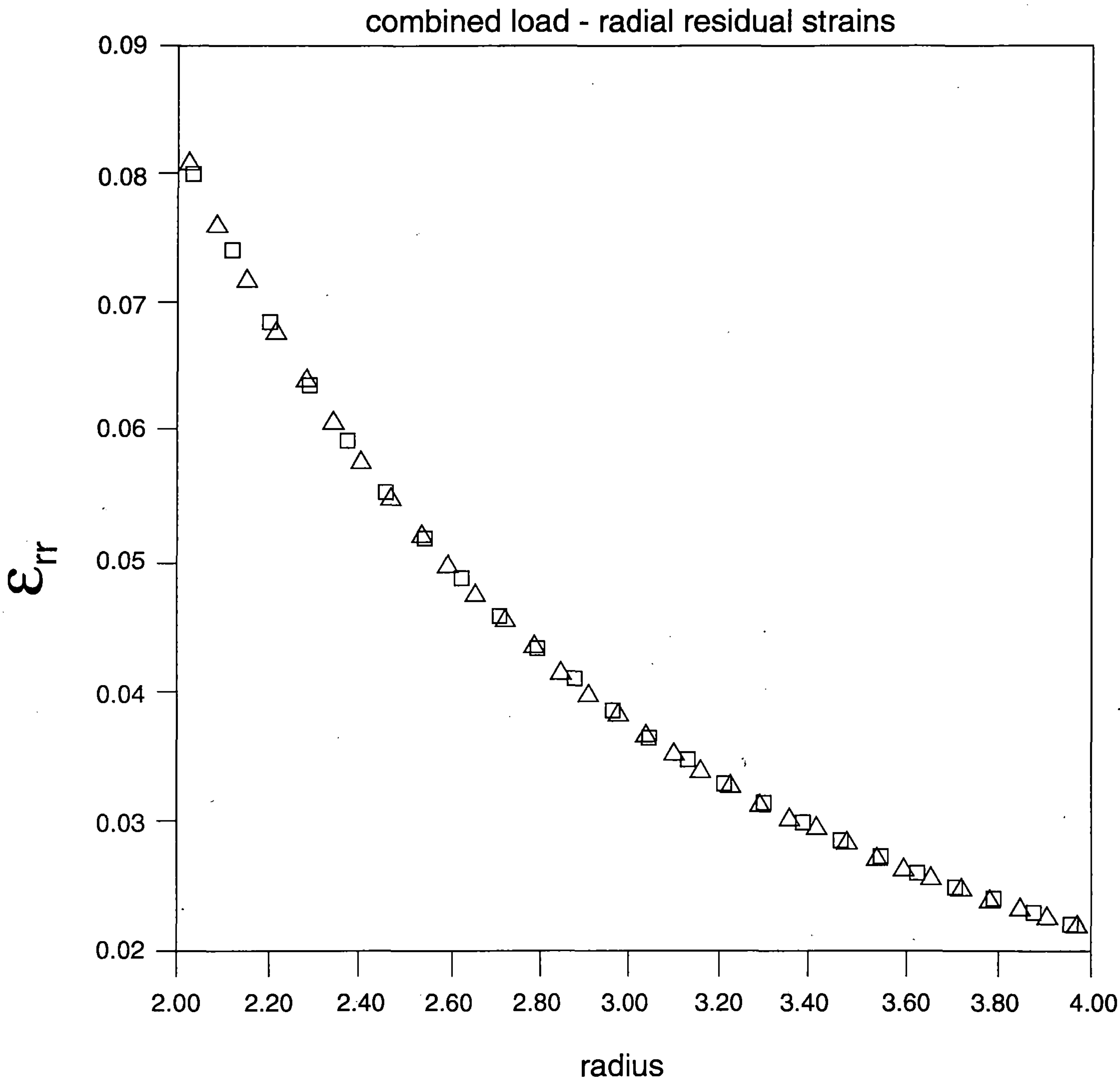

combined load - radial residual strains
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
ε_{rr}
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 28.

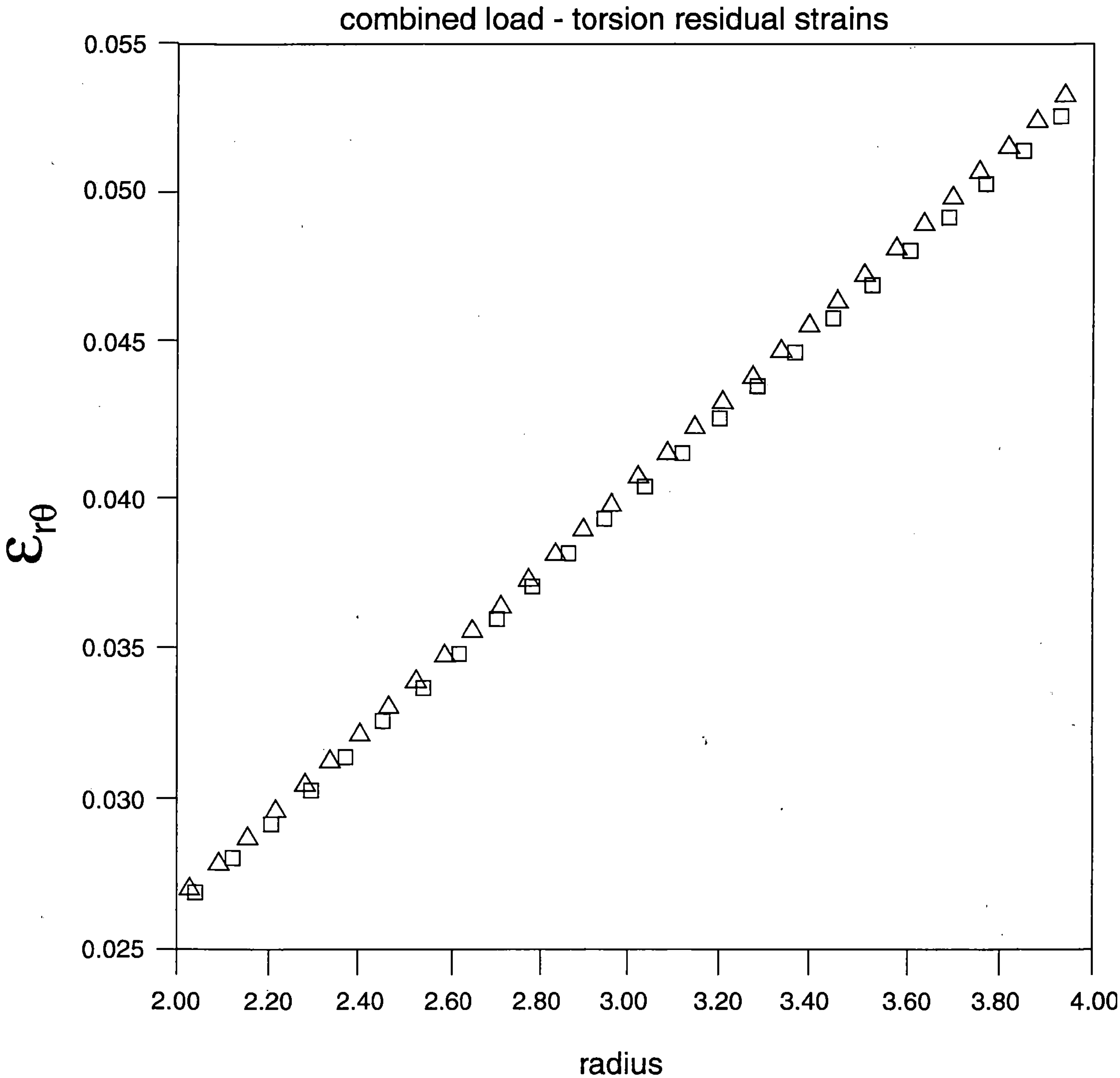

combined load - torsion residual strains
0.055
0.050
0.045
0.040
0.035
0.030
0.025
$\varepsilon_{r\theta}$
2.00
2.20
2.40
2.60
2.80
3.00
3.20
3.40
3.60
3.80
4.00
radius

Figure 29.