REPORT NO. FRA/ORD-78/41

RAIL-WHEEL GEOMETRY ASSOCIATED WITH CONTACT STRESS ANALYSIS

TECHNICAL REPORT NO. 6

BY B. PAUL AND J. HASHEMI

SEPTEMBER 1979

DOCUMENT IS AVAILABLE TO THE PUBLIC
THROUGH THE
NATIONAL TECHNICAL INFORMATION SERVICE

PREPARED FOR THE
FEDERAL RAILROAD ADMINISTRATION

GENERAL DISCLAIMER

This document may be affected by one or more of the following statements

- **This document has been reproduced from the best copy furnished by the sponsoring agency. It is being released in the interest of making available as much information as possible.**
- **This document may contain data which exceeds the sheet parameters. It was furnished in this condition by the sponsoring agency and is the best copy available.**
- **This document may contain tone-on-tone or color graphs, charts and/or pictures which have been reproduced in black and white.**
- **This document is paginated as submitted by the original source.**
- **Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.**

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
FRA/ORD-78/41		
4. Title and Subtitle RAIL-WHEEL GEOMETRY ASSOCIATED WITH CONTACT STRESS ANALYSIS, Technical Report No. 6		5. Report Date September 1979 6. Performing Organization Code
7. Author(s) B. Paul and J. Hashemi		8. Performing Organization Report No. MEAM 79-6
9. Performing Organization Name and Address Department of Mechanical Engineering and Applied Mechanics - University of Pennsylvania Room 111 Towne Building/D3, Philadelphia, PA 19104		10. Work Unit No. (TRAIS) 11. Contract or Grant No. DOT-OS-60144
12. Sponsoring Agency Name and Address U. S. Department of Transportation Federal Railroad Administration Office of Research & Development Washington, DC 20590		13. Type of Report and Period Covered Technical Report 14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

This report records the derivation of a number of results pertaining to wheel and rail geometry that are needed for the analysis of contact stresses and rolling-creepage phenomena. In particular, results utilized in the authors' computer programs COUNTACT (for COUNTerformal contACT problems) and CONFORM (for CONFORMal contact problems) are given.

It is shown how the profile curves specified by engineering drawings for standard wheels and rails may be analyzed to find appropriate parameters needed to express the pertinent equations in the various coordinate systems utilized in contact stress analysis. For arbitrarily selected points of initial contact on the wheel tread and on the railhead, it is shown how to determine the feasibility of such contact, and how to determine the mutual separation of points on the two surfaces. It is also shown how to determine the curve of interpenetration which is used as an initial estimate of the contact patch boundary associated with a given relative approach (due to elastic deformation) of the loaded wheel and rail. The basis of a computer program (MIDSEP) to determine this separation is described.

17. Key Words	18. Distribution Statement
Rail-wheel interaction, contact stress, elasticity, non-Hertzian contact	Document is available through the National Technical Information Service, Springfield, Virginia 22161

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
UNCLASSIFIED	UNCLASSIFIED	27	

Form DOT F 1700.7 (8-72)

Table of Contents

1. INTRODUCTION

The purpose of this report is to provide the derivation for a number of results pertaining to wheel and rail geometry that are needed for the analysis of contact stresses and friction-creepage phenomena. In particular, we wish to record here results that are utilized in the computer programs COUNTACT (Paul and Hashemi, 1977-a,b) and CONFORM (Paul and Hashemi, 1978, 1979).

2. STATEMENT OF THE PROBLEM

The geometry of the wheels and rails for railroad vehicles are usually specified in drawings of so-called profile curves in the plane of symmetry of the wheel or the rail. In Sec. 3 and in Appendix 1 we show how these drawings can be used to provide parameters and equations required for stress analysis purposes.

Rails and wheels can have one or more contact points, referred to as initial contact points, prior to any deformation. The initial contact points depend on the orientation of the wheelset relative to the track. For sufficiently small yaw angles of the wheelset, the initial contact points are located in a midplane.* Only the case of a single initial contact point is discussed in this paper.**

For a given position of the wheel relative to the rail, it is required to determine the initial contact points. This is discussed in Sections 4 and 5.

For a feasible initial contact point, the distance between two points, in the direction of the normal to the rail or wheel at the initial contact point, on the rail and wheel, is required. This distance is called the initial separation, discussed in Sec. 6.

*The plane which passes through the wheel axis and the contact point will be referred to as the midplane.

**For a discussion of how the initial contact points (single or multiple) are influenced by lateral wheelset motions see Cooperrider, et. al. (1975) and Heller and Cooperrider (1977).

The region where the rail and wheel touch, after load is applied, is called contact patch. The determination of an initial estimate for the boundary of the contact patch is discussed in Sec. 7.

3. RAIL AND WHEEL PROFILES

The railhead profile is the curve intersected by the rail surface and a plane normal to the rail axis [see Fig. (1a)]. The wheel profile is the curve of intersection of the wheel surface and a plane through the wheel axis. Standard American wheels and rails are manufactured with profiles consisting exclusively of straight line segments and circular arc segments. To write algebraic expressions for rail and wheel profiles, the following parameters, associated with a coordinate system (x, z) fixed in the wheel (or rail), are needed (see Figs. 2 and 3):

1. The slope a_i and z-intercept b_i of straight line segment i
2. The radius R_i and arc-center coordinates (a_i, b_i) of circular arc segment i.
3. The "end point" coordinates (x_i, y_i) of the right hand end of segment i

In any specific case, information must be provided to calculate the above mentioned parameters (see Appendix 1 for derivation of the parameters associated with the wheel and rail of Figs. 2 and 3).

In the segment i, defined by

$$x_{i-1} < x < x_i \tag{3.1}$$

the "profile equations" (for wheel or rail) are:

$$z = b_i + [R_i^2 - (x - a_i)^2]^{1/2} \text{ (arc segment)} \tag{3.2-a}$$

$$z = a_i x + b_i \text{ (straight segment)} \tag{3.2-b}$$

When necessary, a superscript R or W will be used to distinguish between the coordinates (x^R, z^R) of the rail and the coordinates (x^W, z^W) of the wheel.

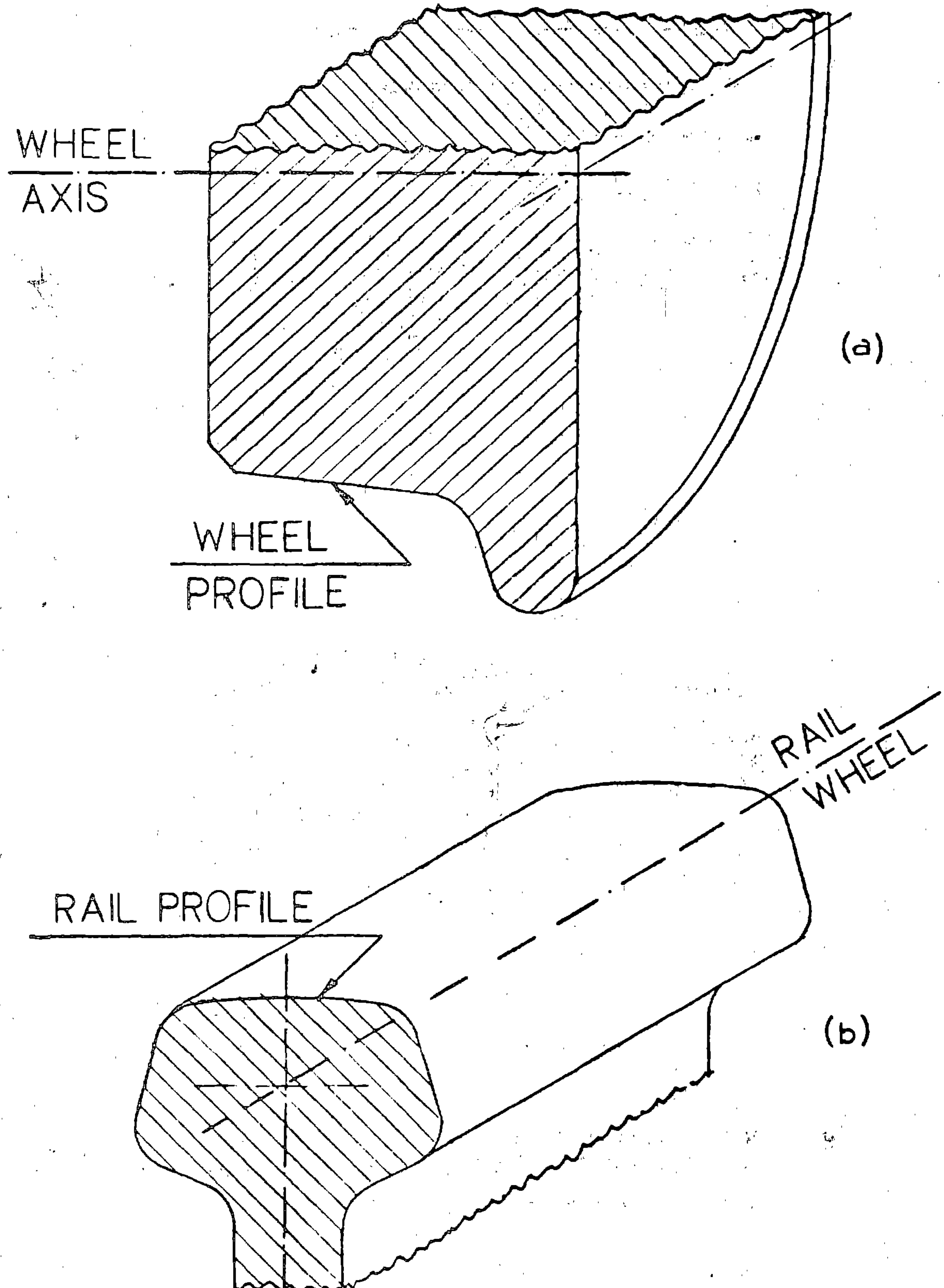
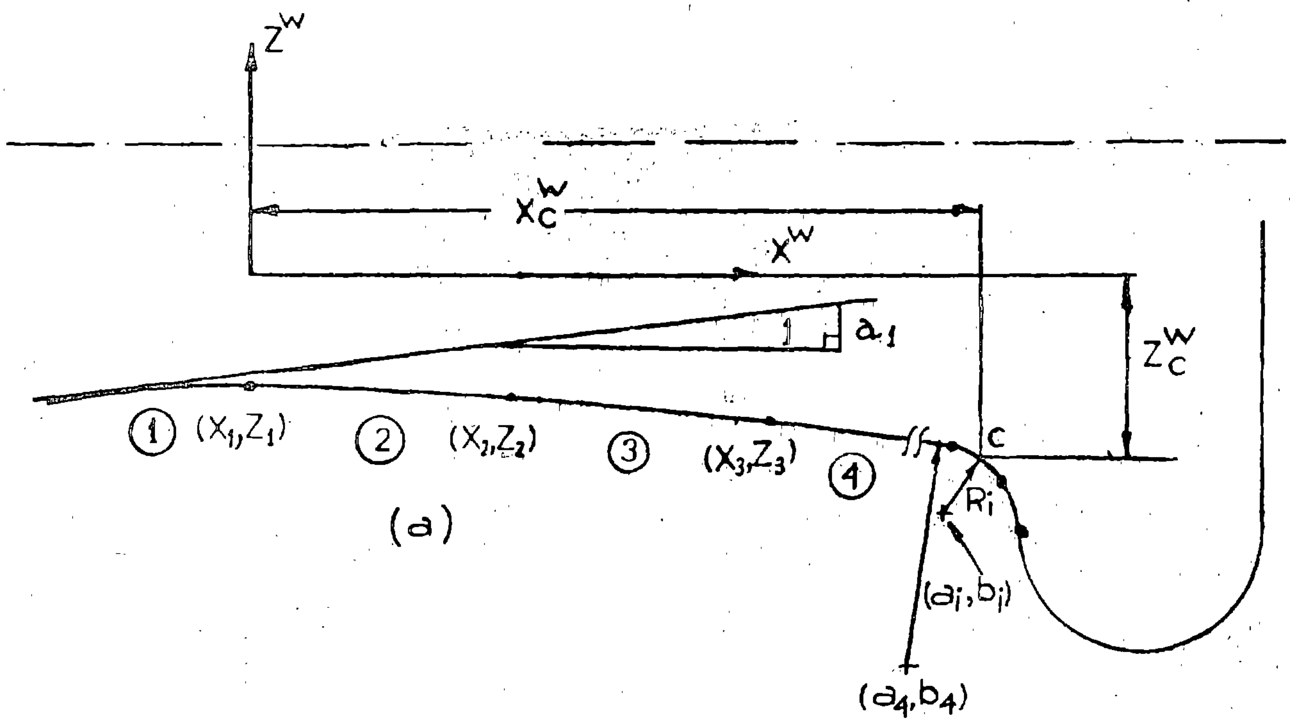


Fig.1 (a) Wheel profile
(b) Rail profile



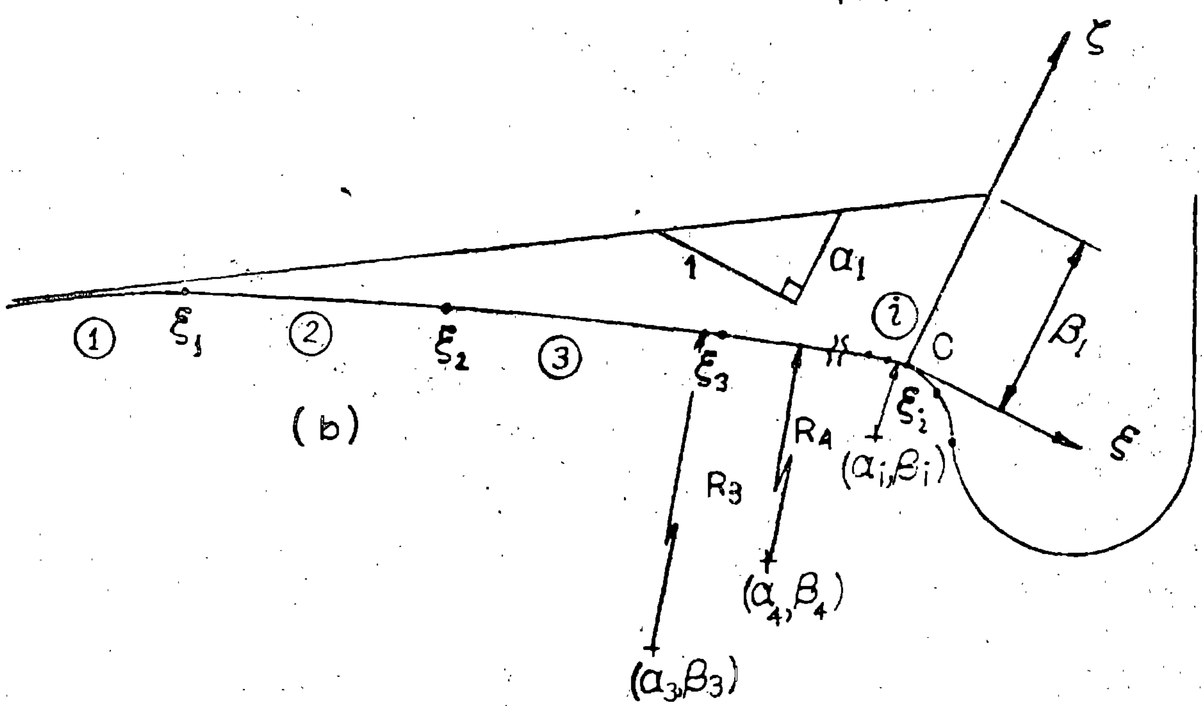


Fig. 2. Wheel profile curves
(a) Parameters referred to wheel reference axes (x^W, z^W)
(b) Parameters referred to global axes (ξ, ζ)

For the usual case of symmetric rails, it is convenient to locate the z-axis parallel to the axis of symmetry of the cross-section (see Fig. 3). The x-axis of the rail is conveniently chosen to be transverse to the rail.

4. FEASIBILITY OF INITIAL CONTACT PATCH

The rail and wheel can have one or more initial contact points prior to deforming under the action of the applied loads--depending upon the orientation of the wheelset relative to the rails. For sufficiently small yaw angles of the wheelset, the initial contact points are located in a plane which passes through the wheel axis and is normal to the rail surface. This plane coincides with the midplane, defined in Sec. 2.

This discussion is limited to the case where there exists a single point of initial contact between wheel and rail.

The initial contact point C on the wheel is determined by its x-coordinate x_C^W (see Fig. 2). Similarly, the initial contact point on the rail is determined by x_C^R (see Fig. 3).

For given locations of C on the wheel and on the rail, let (ξ, η, ζ) be a coordinate system with origin at point C, with the ζ-axis along the common normal, and with the η-axis normal to the midplane, as illustrated in Fig. 4. The initial contact point C is feasible if, and only if, the distance between any two points on the rail and wheel profiles, which have the same values of ξ and η is positive; i.e. when

$$s \equiv \zeta^W(\xi,\eta) - \zeta^R(\xi,\eta) \geq 0 \qquad (4.1)$$

We now need to write the transformation equations between the wheel and rail coordinates (x,z) and the global coordinates (ξ,ζ) for an arbitrary point P. If θ_C is the angle through which the ξ-axis is rotated (positive counterclockwise) with respect to the x-axis and (x_C, z_C) are the coordinates of point C, in the x-z coordinate system, then it follows from Fig. 5 that:

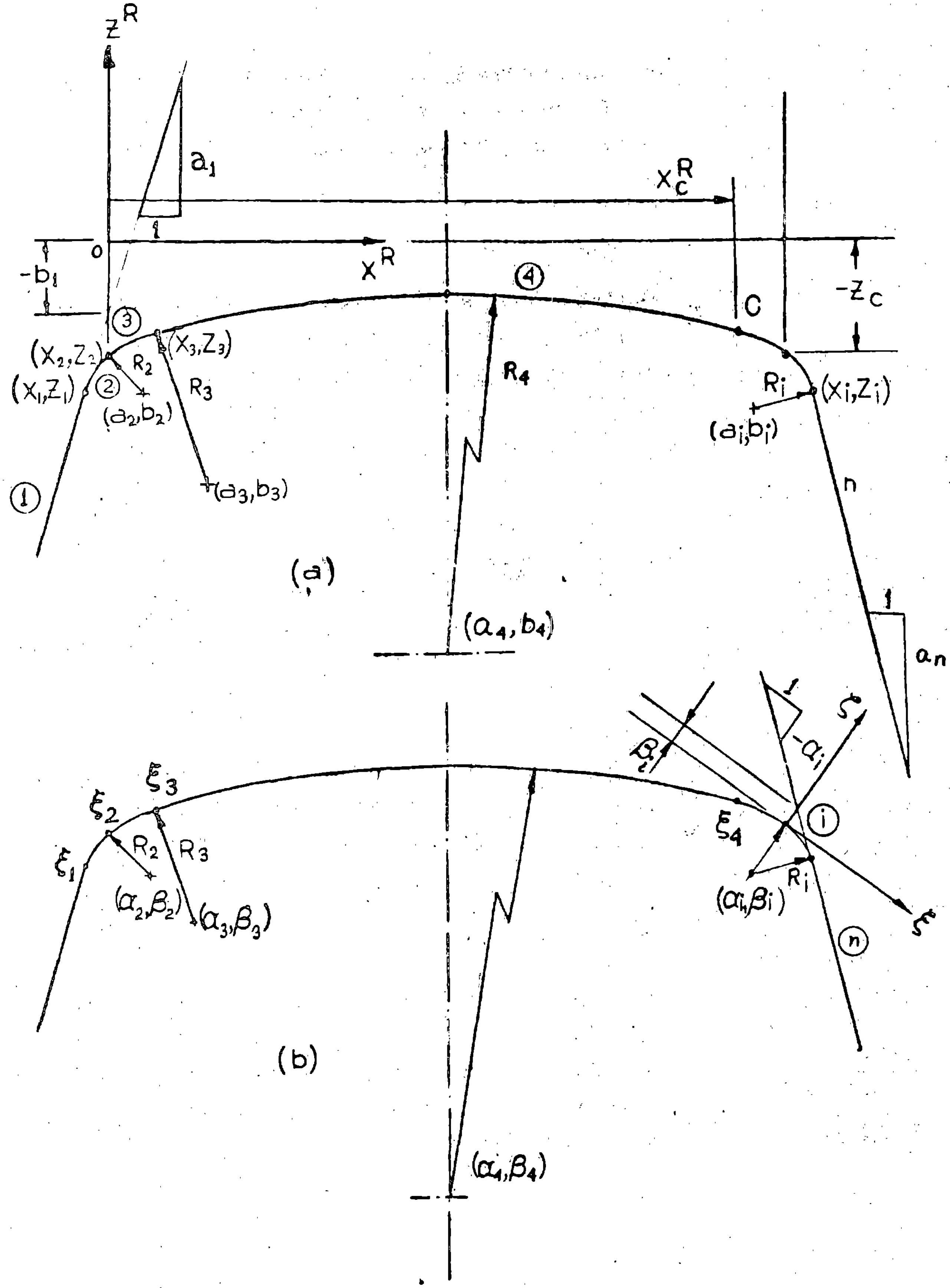


Fig. 3. Rail profile
(a) in rail reference system (x^R, z^R)
(b) in global reference system (ξ, ζ)

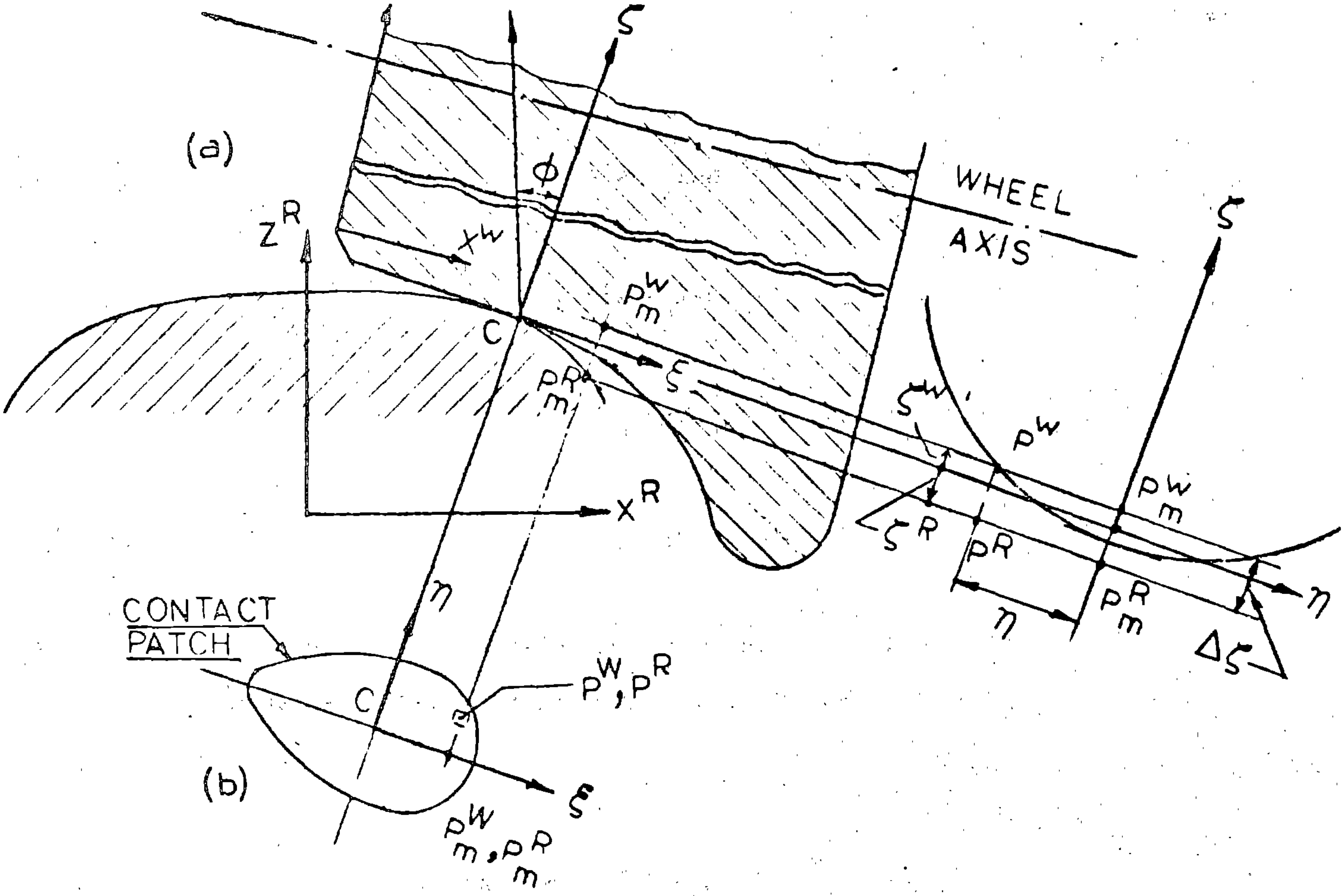


Fig. 4. (a) Cross section through wheel axis and contact point C
(b) Projection of contact patch on plane $\zeta = 0$
(c) Cross section through plane $\xi = 0$

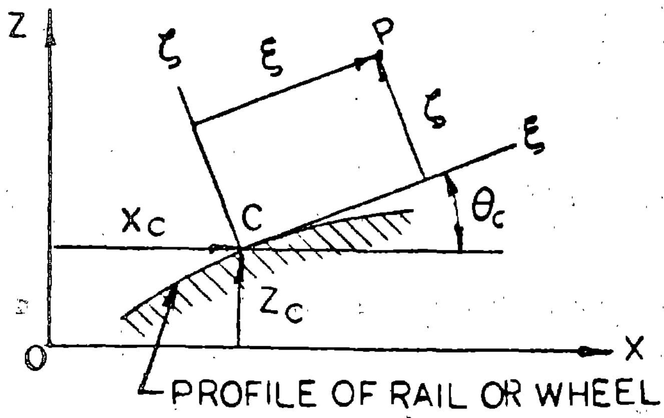


Fig. 5. Transformation from (x,z) to (ξ,ζ) axes

$$\xi = (x-x_c) \cos\theta_c + (z-z_c) \sin\theta_c \quad (4.2\text{-a})$$

$$\zeta = (z-z_c) \cos\theta_c - (x-x_c) \sin\theta_c \quad (4.2\text{-b})$$

or

$$x = x_c + \xi \cos\theta_c - \zeta \sin\theta_c \quad (4.3\text{-a})$$

$$z = z_c + \xi \sin\theta_c + \zeta \cos\theta_c \quad (4.3\text{-b})$$

Substituting Eqs. (4.3) into Eqs. (3.2) and solving for ζ, we obtain the transformed equations (4.5-a,b) of the profile, in the range

$$\xi_{i-1} < \xi < \xi_i \quad (4.4)$$

$$\zeta = \begin{cases} \beta_i + [R_i^2 - (\xi-\alpha_i)^2]^{1/2} & \text{(arc segments)} \quad (4.5\text{-a}) \\ \alpha_i \xi + \beta_i & \text{(straight segments)} \quad (4.5\text{-b}) \end{cases}$$

where:

$$\alpha_i = \begin{cases} (a_i-x_c) \cos\theta_c + (b_i-z_c) \sin\theta_c & \text{(arc)} \quad (4.6\text{-a}) \\ \dfrac{a_i \cos\theta_c - \sin\theta_c}{a_i \sin\theta_c + \cos\theta_c} & \text{(straight)} \quad (4.6\text{-b}) \end{cases}$$

similarly;

$$\beta_i = \begin{cases} (b_i-z_c) \cos\theta_c - (a_i-x_c) \sin\theta_c & \text{(arc)} \quad (4.7\text{-a}) \\ (a_i x_c + b_i-z_c)/(\cos\theta_c + a_i \sin\theta_c) & \text{(straight)} \quad (4.7\text{-b}) \end{cases}$$

and

$$\xi_i = (x_i-x_c) \cos\theta_c + (z_i-z_c) \sin\theta_c \quad (4.8)$$

ξ_i is the segment end point abscissa in the (ξ,ζ) system.

For a given value of x_c, the appropriate segment number i is found from the range restriction

$$x_{i-1} < x_c < x_i \tag{4.9}$$

and the corresponding value of z_c is given by Eqs. (3.2) in the form

$$z_c = \begin{cases} b_i + [R_i^2 - (x_c - a_i)^2]^{1/2} & \text{(arc)} \quad (4.10\text{-a}) \\ a_i \; x_c + b_i & \text{(straight)} \quad (4.10\text{-b}) \end{cases}$$

From Figure 5, we see that the profile slope at point C is given by

$$\left(\frac{dz}{dx}\right)_c = \tan \theta_c \tag{4.11}$$

Hence the angle θ_c is given by

$$\theta_c = \tan^{-1} \left[\frac{dz}{dx}\right]_{x=x_c} = \tan^{-1} \begin{cases} \dfrac{-(x_c - a_i)}{[R_i^2 - (x_c - a_i)^2]^{1/2}} & \text{(arc)} \quad (4.12\text{-a}) \\ a_i & \text{(straight)} \quad (4.12\text{-b}) \end{cases}$$

Therefore, with (a_i, b_i), R_i, x_i and x_c given for the rail (wheel), $\zeta^R(\zeta^W)$ is obtained, for a given value of ξ , by the following procedure:

(a) calculate z_c and θ_c using Eqs. (4.10) and (4.12) respectively.

(b) From Eqs. (4.6), (4.7) and (4.8), calculate (α_i, β_i) and ξ_i respectively.

(c) Finally, calculate ζ^R (ζ^W) from Eqs. (4.5).

5. OUTLINE OF COMPUTER PROGRAM MIDSEP

Based on the above equations and procedures, a FORTRAN program* called MIDSEP (MIDplane SEParation) was written to provide the following as output:**

(a) Transformed rail and wheel parameters: α_i, β_i, ξ_i (see Figs. 2 and 3)

(b) Value of the separation function $\Delta\zeta = \zeta^W - \zeta^R$ on the plane $\eta = 0$, at equally spaced values of ξ.

The program is organized as shown in Fig. 6. A brief description of each program block follows.

Main Program (MIDSEP):

The purpose of the main program is to manage input and output, to call appropriate subprograms as needed, and to interlink the various components needed for overall program logic. It calls upon subroutines RAILO and WHEELO to calculate z_c, θ_c, α_i, β_i, ξ_i, for rail and wheel. Then it will continue to calculate ζ^R and ζ^W for equal increments of ξ, by calling subroutines RAIL and WHEEL. Finally it calculates $\Delta\zeta = \zeta^W - \zeta^R$.

Subroutine RAILO: Calculates the z-coordinate of the initial contact point on the rail, and the transformed parameters (α_i, β_i, ξ_i) needed for the rail.

Subroutine WHEELO: Calculates the z-coordinate of the initial contact point on wheel and the transformed parameters (α_i, β_i, ξ_i) needed for the wheel.

Subfunction RAIL: Calculates the ζ^R for any ξ; i.e. the rail profile in the global coordinate system (ξ,ζ) of the midplane.

Subfunction WHEEL: Calculates the ζ^W for any ξ; i.e. the wheel profile in the global coordinate system (ξ,ζ) of the midplane.

A numerical example which fully illustrates the use of Program MIDSEP is given in Appendix 2 of Paul and Hashemi [1978].

---

*A complete User's Manual for program MIDSEP, including the FORTRAN listing is given in the "User's Manual for Program CONFORM", Paul and Hashemi [1978].

**The output described here serves as input for the calculations of Section 6.

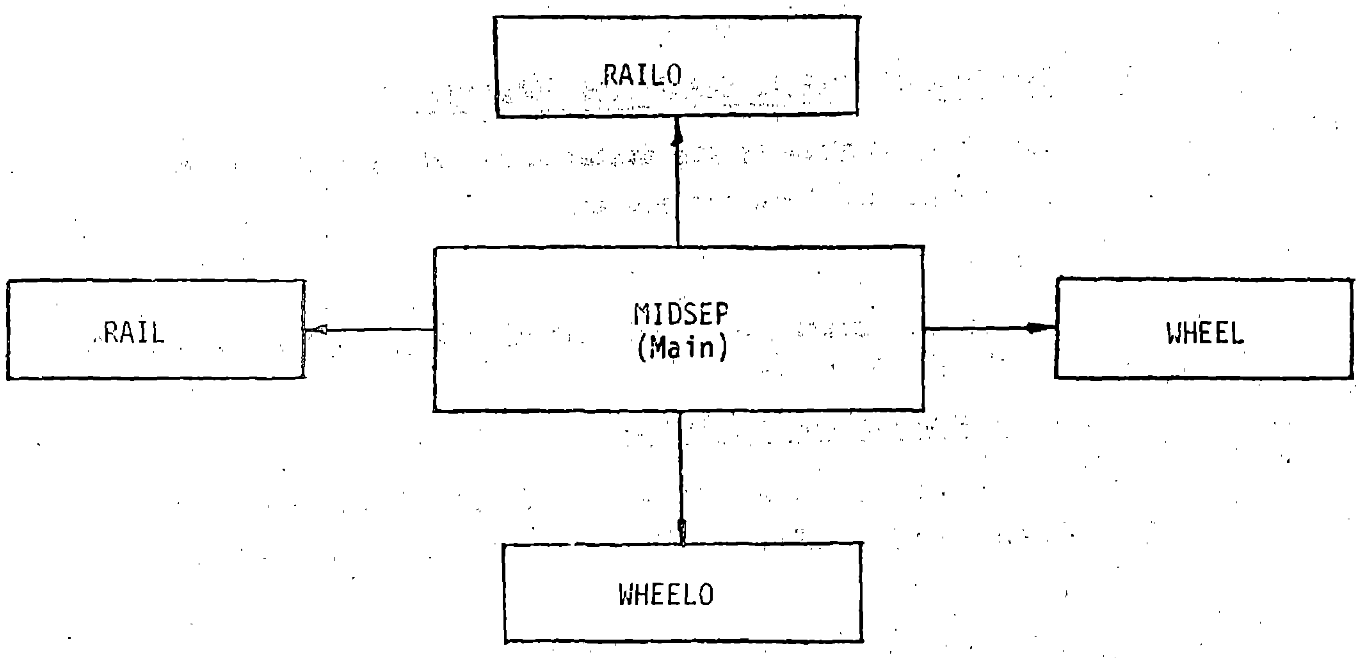


Fig. 6. Organization of program MIDSEP. Arrows point from calling to called program.

6. NUMERICAL DETERMINATION OF INITIAL SEPARATION FUNCTION

The initial separation function is the distance between two points in the ζ-direction. It is obtained from the equation

$$\Delta\zeta = \zeta^W - \zeta^R \tag{6.1}$$

where ζ^W is the wheel shape function and ζ^R is the rail shape function, both corresponding to the same value of ξ.

6.1 Rail Shape Function and Unit Normal Vector

The rail is a cylindrical surface. Therefore, in the global coordinate system (ξ,η,ζ) its shape function will be the same as its profile function given by equations (4.5), i.e.

$$\zeta^R = \begin{cases} \beta_i + [R_i^2 - (\xi - \alpha_i)^2]^{1/2} & \text{(arc)} \\ \alpha_i \xi + \beta_i & \text{(straight)} \end{cases} \tag{6.2}$$

where α_i, β_i and ξ_i are given by Eqs. (4.6-4.8). The components of the unit vector $\mathbf{m}$, normal to the rail surface, are obtained as follows:

$$\mathbf{m} = (n_\xi, n_\eta, n_\zeta) = (\frac{\partial F}{\partial \xi}, 0, \frac{\partial F}{\partial \zeta})/N \tag{6.3}$$

where

$$N = [(\frac{\partial F}{\partial \xi})^2 + (\frac{\partial F}{\partial \zeta})^2]^{1/2} \tag{6.4}$$

and

$$F(\xi,\zeta) = \begin{cases} \beta_i + [R_i^2 - (\xi-\alpha_i)^2]^{1/2} - \zeta & \text{(arc)} \\ \alpha_i \xi + \beta_i - \zeta & \text{(straight)} \end{cases} \tag{6.5}$$

Therefore,

$$\frac{\partial F}{\partial \xi} = \begin{cases} \dfrac{-(\xi-\alpha_i)}{[R_i^2-(\xi-\alpha_i)^2]^{1/2}} & \text{(arc)} \\ \alpha_i & \text{(straight)} \end{cases} \tag{6.6}$$

$$\frac{\partial F}{\partial \zeta} = -1$$

$$N = \begin{cases} \dfrac{R_i}{[R_i^2-(\xi-\alpha_i)^2]^{1/2}} & \text{(arc)} \\ [1+\alpha_1^2]^{1/2} & \text{(straight)} \end{cases} \tag{6.7}$$

$$n_\xi = \begin{cases} (\xi-\alpha_i)/R_i & \text{(arc)} \\ \alpha_i[1+\alpha_i^2]^{-1/2} & \text{(straight)} \end{cases} \tag{6.8}$$

$$n_\zeta = \begin{cases} (\zeta-\beta_i)/R_i & \text{(arc)} \\ [1+\alpha_i^2]^{-1/2} & \text{(straight)} \end{cases} \tag{6.9}$$

6.2 Wheel Shape Function

The wheel is a surface of revolution with nominal radius R^W (the distance from the wheel axis to the origin O). Its surface equation in the wheel coordinate system (x^W, y^W, z^W) in the domain $x_{i-1} \leq x \leq x_i$ is (see Fig. 7):

$$(R^W - z^W)^2 - (\rho^2 - y^2) = 0 \tag{6.10}$$

where

$$\rho = R^W - z_0^W = \begin{cases} R^W - (\sqrt{R_i^2 - (x-a_i)^2} + b_i) & \text{(arc)} \\ R^W - (a_i x + b_i) & \text{(straight)} \end{cases} \tag{6.11}$$

Therefore,

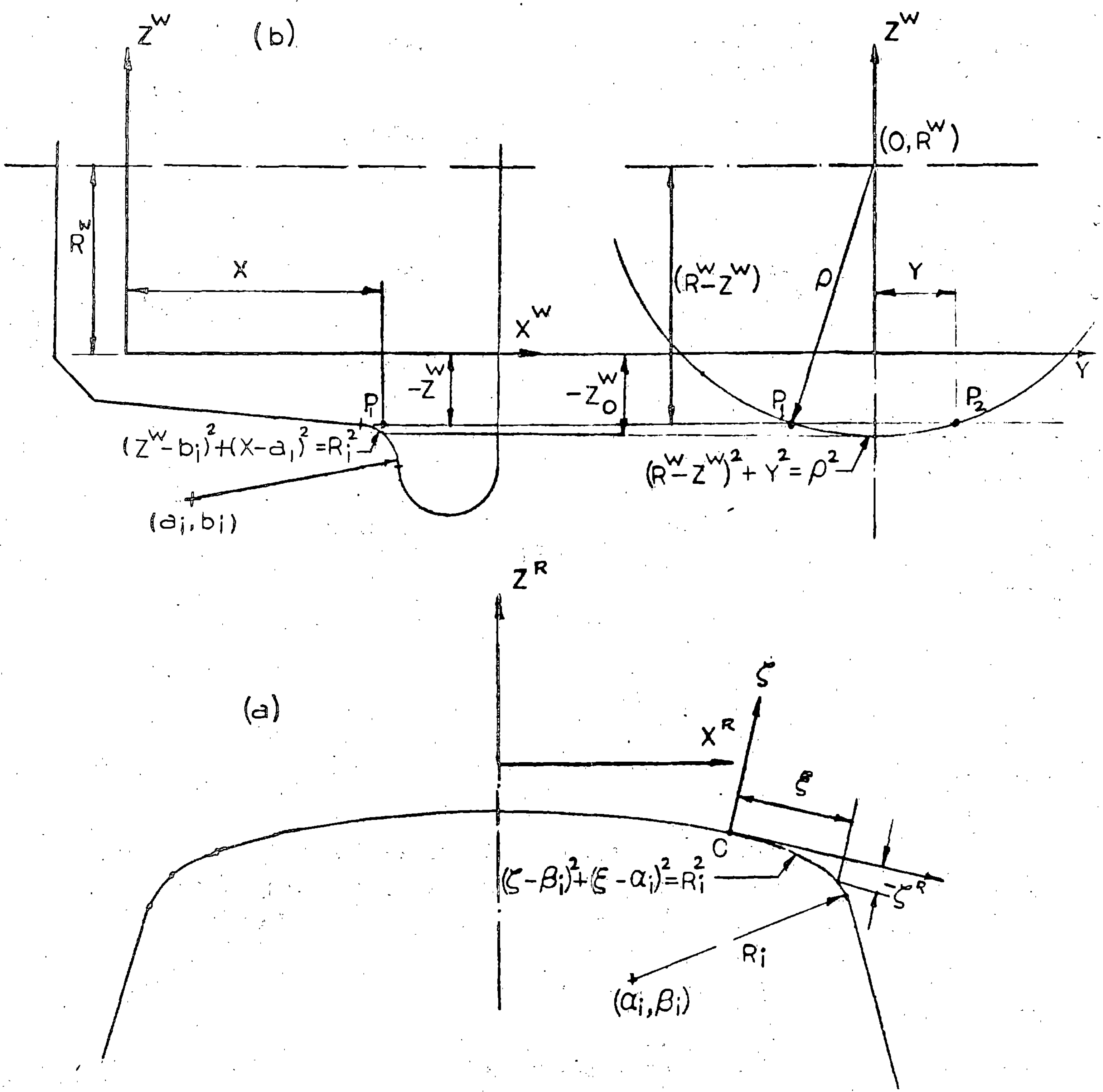


Fig. 7 Rail and wheel profile functions

$$z^W = \begin{cases} R^W - \{[R^W - (\sqrt{R_i^2 - (x-a_i)^2} + b_i)]^2 - y^2\}^{1/2} & \text{(arc)} \\ \\ R^W - \{[R^W - (a_i x + b_i)]^2 - y^2\}^{1/2} & \text{(straight)} \end{cases} \quad (6.12)$$

We may express z^W in terms of ξ and η if we note that $y \equiv \eta$, and x is given by Eq. (4.3-a) .

From Fig. 7 , it is apparent that

$$F(\xi,\eta,\zeta^W) \equiv (z^W - R^W)^2 - (\rho^2 - \eta^2) = 0 \quad (6.13)$$

where $F(\xi,\eta, \zeta^W)$ is, by definition, the "wheel shape function."

Our objective will be to find ζ^W for given values of ξ and η . However, we cannot solve Eq. (6.13) directly because ζ^W appears in it implicitly in a highly nonlinear fashion. Therefore, we will use the Newton-Raphson iterative method [see Hamming, 1971, pp. 49-51] which requires that we guess at an initial value of ζ^W and then find an improved value

$$\zeta^W + \Delta\zeta^W \quad (6.14)$$

where the incremental correction is given by

$$\Delta\zeta^W = - F(\zeta^W)/ (\frac{dF}{d\zeta^W}) \quad (6.15)$$

where F is the wheel shape function defined by Eq. (6.13). Since ζ^W enters Eq. (6.13) through the terms z^W and ρ , and ρ is a function of x, we can evaluate $dF/d\zeta^W$ in the form

$$\frac{dF}{d\zeta^W} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial \zeta} + \frac{\partial F}{\partial \rho}\frac{\partial \rho}{\partial x}\frac{\partial x}{\partial \zeta} \quad (6.16)$$

The procedure is as follows:

a. Guess a value for ζ^W (for example, use Eq. (4.5) to find ζ^W corresponding to $\eta = 0$).

b. Find $F(\zeta^W)$ from equation (6.13).

c. Find $\frac{\partial x}{\partial \zeta}$, $\frac{\partial z}{\partial \zeta}$, and $\frac{\partial \rho}{\partial x}$ from Eqs. (4.3) and (6.11), in the form

$$\frac{\partial x}{\partial \zeta} = - \sin \theta_c \tag{6.17}$$

$$\frac{\partial z}{\partial \zeta} = \cos \theta_c \tag{6.18}$$

$$\frac{\partial \rho}{\partial x} = \begin{cases} \dfrac{(x-a_i)}{\left[R_i^2-(x-a_i)^2\right]^{1/2}} & \text{(arc)} \\ -a_i & \text{(straight)} \end{cases} \tag{6.19}$$

d. Find ρ from Eq. (6.11), then $\frac{\partial F}{\partial \rho}$ by differentiating Eq. (6.13), i.e.

$$\frac{\partial F}{\partial \rho} = -2\rho \tag{6.20}$$

e. Find $\frac{dF}{d\zeta}$ from Eq. (6.16)

f. Find $\Delta\zeta$ from Eq. (6.15)

g. Find the new value for ζ from;

$$\zeta_{new} = \zeta_{old} + \Delta\zeta \tag{6.21}$$

h. Repeat steps a through g, with the new value of ζ, until a desired tolerance is reached for $|\Delta\zeta|$.

With ζ^R and ζ^W determined at any point of the ξ-η plane, the separation is calculated from

$$s = \zeta^W - \zeta^R \tag{6.22}$$

Subroutine INSEP

The subroutine INSEP--which stands for "INITIAL SEParation" -- has been based on the analysis just discussed. Its purpose is to supply the initial separation between rail and wheel (and the unit normal components only for program CONFORM), to the calling program. It calls upon subprogram RAIL and MIDWEL to find ζ^R and ζ^W when $\eta = 0$; then INSEP calculates $s = \zeta^W - \zeta^R$. For $\eta \neq 0$, ζ^R remains the same and INSEP calls upon WHEEL to calculate ζ^W.

The organization of subroutine INSEP is shown in Fig. 8 , where the various subprograms have the following purposes.*

Subroutine <u>INSEP</u>: calculates s, the initial separation between rail and wheel

Subroutine <u>RAIL</u>: calculates ζ^R for any given ξ (also components of unit normal vector when used with CONFORM)

Subroutine <u>MIDWEL</u>: calculates ζ^W when $\eta = 0$

Subroutine <u>WHEEL</u>: calculates ζ^W when $\eta \neq 0$

Subroutine <u>WHEELO</u>: calculates z^W and dz^W/dx.

7. INITIAL CANDIDATE CONTACT BOUNDARY

An initial estimate of the contact patch is essential for the numerical solution of contact stress problems. The projection on the ξ-ζ plane of the intersection curve of the two rigid surfaces, obtained by their interpenetration through a distance δ, along ζ-axis is a good first approximation to the actual region of contact for a rigid body approach δ. This region is called the initial <u>candidate contact patch</u> and its boundary is called the <u>interpenetration curve</u> (see Fig. 9). The interpenetration curve is given by

$$\zeta^W - \zeta^R \equiv f(\xi,\eta) = \delta \tag{7.1}$$

In general, it will not be possible to get an analytical expression for $f(\xi,\eta)$. Therefore, we use ξ as the independent variable, and find the value of $\eta(=y)$ which satisfies Eq. (7.1) by means of the following procedure.

<u>Procedure to find interpenetration curve</u>

a. Calculate ζ^R from Eq. (6.2) for a <u>given</u> ξ.

b. Calculate ζ^W from Eq. (7.1) as;

$$\zeta^W = \delta + \zeta^R \tag{7.2}$$

c. Calculate x from Eq. 4.3-a)) using ξ and ζ^W.

---

*Note that the subprograms named RAIL, WHEEL and WHEELO used with Subroutine INSEP are different from similarly named subprograms used with program MIDSEP (Sec. 5). A complete FORTRAN listing of INSEP and all its associated subroutines is included with the listing of CONFORM in Paul and Hashemi [1978].

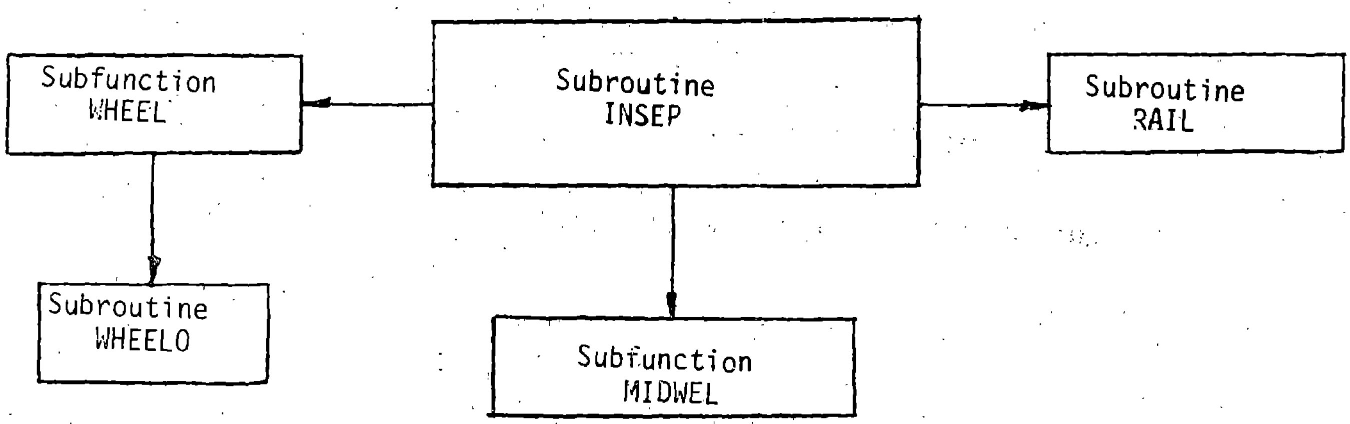


Fig. 8 Organization of subroutine INSEP

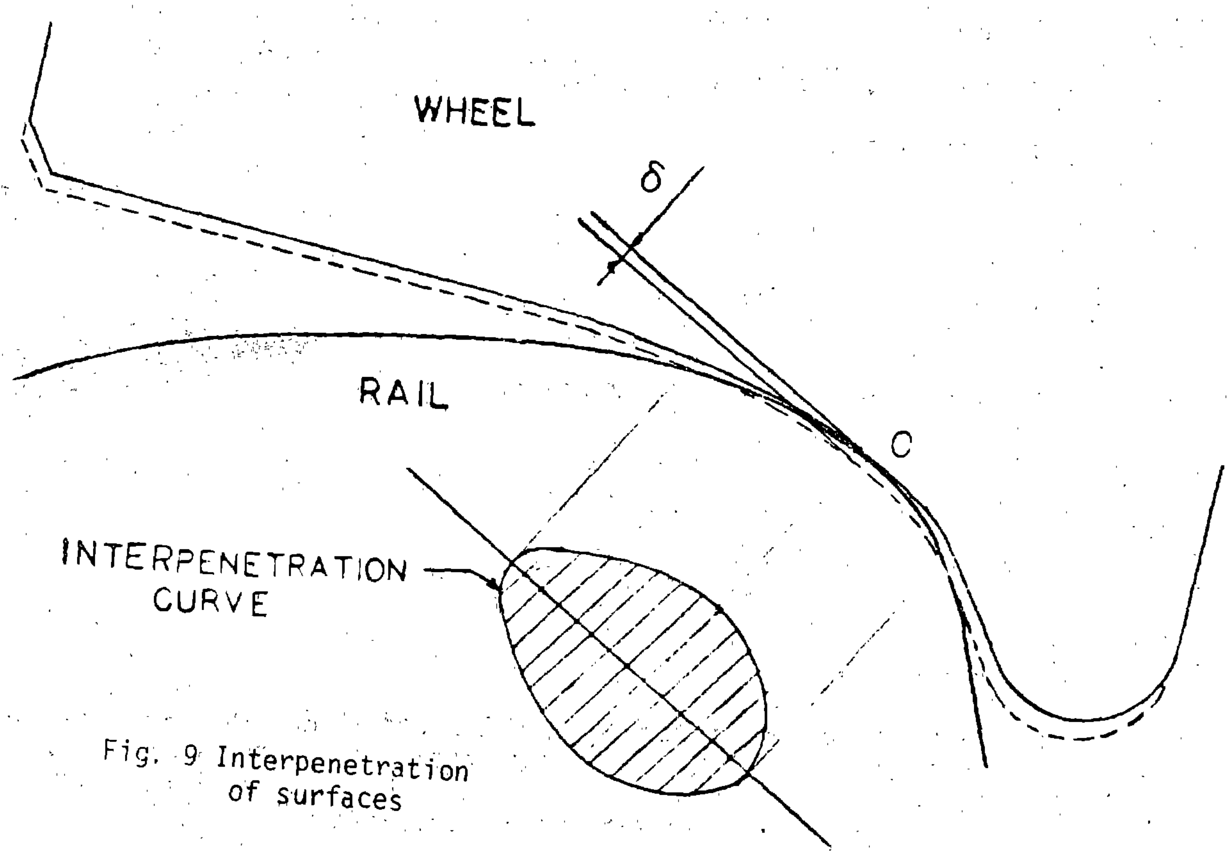


Fig. 9 Interpenetration of surfaces

d. Calculate z^W from Eq. (4.3-b) and $\rho(x)$ from Eq. (6.11).

e. Calculate $\eta(=y)$ by solving (Eq. 6.10)

$$y = \eta = \pm \left[\rho^2(-z^W - R^W)\right]^{1/2} \tag{7.3}$$

With η calculated for any value of ξ, the interpenetration curve is determined. Based on the above procedure, a FORTRAN program called INTERPEN was written to determine the interpenetration curve for any given rigid body displacement. For Step a, INTERPEN calls the same subroutine RAIL, as that called by program MIDSEP. For step e, INTERPEN calls a subroutine named YRW.

A complete User's Manual and sample problem for Program INTERPEN will be found in Appendix 3 of Paul and Hashemi [1978]. A FORTRAN listing of INTERPEN and its associated subroutines is also given in that reference.

REFERENCES

Cooperrider, N. K., E. H. Law, R. Hall, P. S. Kadala, and J. M. Tuten," "Analytical and Experimental Determination of Nonlinear Wheel/Rail Geometrical Constraints," Rept. No. FRA-OR&D 76-244 (PB 252290), (1975).

Hamming, R. W., Introduction to Applied Numerical Analysis, McGraw-Hill, NY, (1971).

Heller, R. and N. K. Cooperrider, "User's Manual for Asymmetric Wheel/Rail Characterization Program," Rept. No. FRA/OR&D-78/05, (PB 279707/AS) (1977).

Paul, B., and Hashemi, J., "An Improved Numerical Method for Counterformal Contact Stress Problems," Technical Report No. 3, FRA/ORD-78/26, Contract DOT-OS-60144, PB 286228/AS, (1977-a).

Paul, B., and Hashemi, J., "User's Manual for Program COUNTACT COUNTerformal contact stress problems," Technical Report No. 4, FRA/ORD-78/27, Contract DOT-OS-60144, PB 286097/AS, (1977-b).

Paul, B., and Hashemi, J., "User's Manual for Program CONFORM (CONFORMal contact stresses between wheels and rails", Technical Report No. 5, FRA/ORD-78/40, Contract DOT-OS-60144, PB 288927/AS, (1978).

Paul, B. and Hashemi, J., "Numerical Determination of Contact Pressures Between Closely Conforming Wheels and Rails," Technical Report No. 8, FRA/ORD-79/41, Contract DOT-OS-60144, (1979).

APPENDIX A. SMOOTH RAIL AND WHEEL PROFILE

It is desired to be able to write the algebraic equation of each segment of a wheel or rail profile from the information supplied on an engineering drawing. These drawings usually show circular arcs which are joined smoothly to straight lines.

Smooth transition from one segment to a following segment in a rail or wheel profile may occur in the following ways:

a. From a straight line segment to a circular arc segment;

b. From a circular arc segment to another circular arc segment;

c. From a circular arc segment to a straight segment.

In most cases the slope of straight segments, radius of a circular arc segment, and coordinates of segment end points are given.

Smooth Transition from a Straight Line to a Circular Arc Segment [Fig. A-1(a)]

If the slope of the line is a_{i-1}, its y-intercept is b_{i-1}. Let x_{2-1} be the greatest abscissa on segment (i-1), also let R_i be the radius of the circular arc segment i with the end point abscissa x_i [see Fig. A-1(a)]. Then;

$$AC = R_i \sin\theta = \frac{a_{i-1}}{\sqrt{1+a_{i-1}^2}} R_i \tag{A-1}$$

$$AB = R_i \cos\theta = \frac{1}{\sqrt{1+a_{i-1}^2}} R_i \tag{A-2}$$

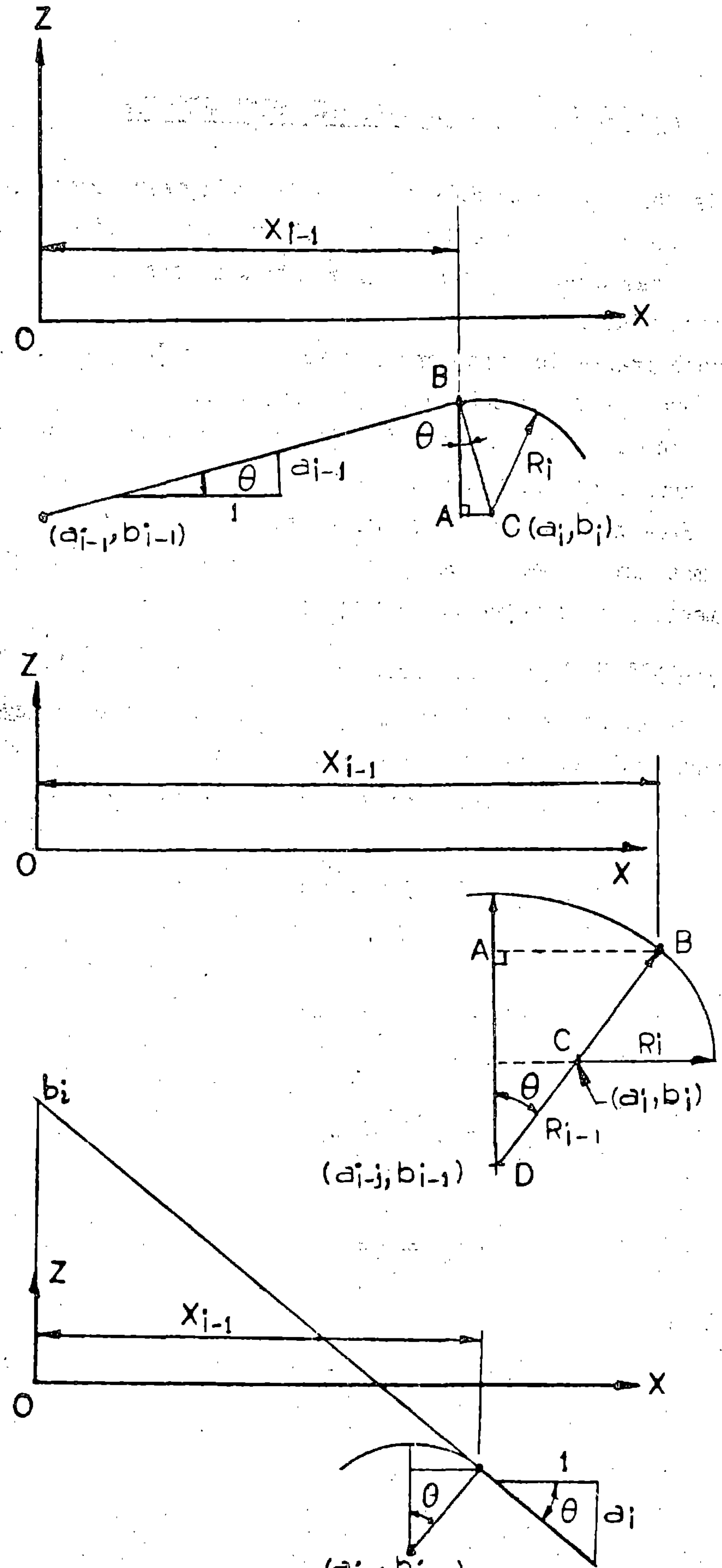
or

$$a_i = x_{i-1} + \frac{a_{i-1}}{\sqrt{1+a_{i-1}^2}} R_i \tag{A-3}$$

$$b_i = z_{i-1} - \frac{R_i}{\sqrt{1+a_{i-1}^2}} \tag{A-4}$$

where

$$z_{i-1} = a_{i-1}\, x_{i-1} + b_{i-1} \tag{A-5}$$



(a) Transition from straight line to circular arc

(b) Transition from circular arc to a different circular arc

(c) Transition from a circular arc to a straight line

Fig. A-1 Profile transitions

Smooth Transition from a Circular Arc Segment to Another Circular Arc Segment [Fig. A-1(b)

If we are given the arc center coordinates (a_{i-1}, b_{i-1}), the radius R_{i-1}, the greatest end point coordinate x_{i-1} of segment i-1, and R_i then: radius of arc segment i is given, then:

$$EC = (R_{i-1} - R_i) \sin \theta = (R_{i-1} - R_i) \frac{x_{i-1} - a_{i-1}}{R_{i-1}} \qquad \text{(A-6)}$$

$$ED = (R_{i-1} - R_i) \cos \theta = (R_{i-1} - R_i)[-(\frac{b_{i-1} - z_{i-1}}{R_{i-1}})] \qquad \text{(A-7)}$$

$$a_i = a_{i-1} + (1 - \frac{R_i}{R_{i-1}})(x_{i-1} - a_{i-1}) \qquad \text{(A-8)}$$

$$b_i = b_{i-1} - (1 - \frac{R_i}{R_{i-1}})(b_{i-1} - z_{i-1}) \qquad \text{(A-9)}$$

where

$$z_{i-1} = b_{i-1} + [R_{i-1}^2 - (x_{i-1} - a_{i-1})^2]^{1/2} \qquad \text{(A-10)}$$

Smooth Transition from a Circular Arc Segment to a Straight Line Segment [Fig.A-1(c)]

If we are given the arc center coordinates (a_{i-1}, b_{i-1}), radius R_{i-1}, and end point abscissa x_{i-1} of segment i-1, then the slope of the straight segment is:

$$a_i = -\tan \theta = -\frac{x_{i-1} - a_{i-1}}{z_{i-1} - b_{i-1}} \qquad \text{(A-11)}$$

and

$$b_i = z_{i-1} - a_i x_{i-1} \qquad \text{(A-12)}$$

where

$$z_{i-1} = b_{i-1} + [R_{i-1}^2 - (x_{i-1} - a_{i-1})^2]^{1/2} \qquad \text{(A-13)}$$

LIST OF RELATED REPORTS AND PUBLICATIONS

A. FRA Technical Reports (Available from National Technical Information Service)

A1. Paul, B., "A Review of Rail-Wheel Contact Stress Problems," Technical Report No. 1, April 1975, FRA/ORD-76 141, PB 251238/AS, Contract DOT-OS-40093.

A2. Woodward, W., and Paul, B., "Contact Stresses for Closely Conforming Bodies - Application to Cylinders and Spheres," Technical Report No. 2, December 1976, DOT/TST/77-48, PB 271033/AS, Contract DOT-OS-40093.

A3. Paul, B., and Hashemi, J., "An Improved Numerical Method for Counterformal Contact Stress Problems," Technical Report No. 3, July 1977, FRA/ORD-78/26, Contract DOT-OS-60144, PB 286228/AS.

A4. Paul, B., and Hashemi, J., "User's Manual for Program COUNTACT COUNTer-formal contACT stress problems ", Technical Report No. 4, September 1977, FRA/ORD-78/27, Contract DOT-OS-60144. PB 286097/AS

A5. Paul, B., and Hashemi, J., "User's Manual for Program CONFORM (CONFORMal contact stresses between wheels and rails ", Technical Report No. 5, June 1978, FRA/ORD-78/40, Contract DOT-OS-60144, PB 288927/AS.

A6. Paul, B., and Hashemi, J., "Rail-Wheel Geometry Associated with Contact Stress Analysis," Technical Report No. 6, September 1979, FRA/ORD-78/41. Contract DOT-OS-60144.

A7. Paul, B., and Hashemi, J., "Contact Stresses in Bodies with Arbitrary Geometry, Applications to Wheels and Rails," Technical Report No. 7, April 1979, FRA/ORD/79-23, Contract DOT-OS-60144.

A8. Paul, B., and Hashemi, J., "Numerical Determination of Contact Pressures Between Closely Conforming Wheels and Rails", Technical Report No. 8. July, 1979, FRA/ORD-79/41, Contract DOT-OS-60144 .

B. Related Papers Published in Various Journals and Proceedings

B1. Singh, K. P., and Paul, B., "A Method for Solving Ill-Posed Integral Equation of the First Kind," Computer Methods in Applied Mechanics and Engineering, Vol. 2, 1973, 339-348.

B2. Singh, K. P., and Paul, B.,"Numerical Solution of Non-Hertzian Elastic Contact Problems," Journal of Applied Mechanics, Vol. 41, Trans. of ASME, Series E, Vol. 96, June 1974, pp. 484-490.

B3. Singh, K. P., and Paul, B., "Stress Concentration in Crowned Rollers," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 97, No. 3, 1975, pp. 990-994.

B4. Paul, B., K. P. Singh., and Woodward, W., "Contact Stresses for Multiply-Connected Regions--The Case of Pitted Spheres," Proceedings of the Symposium on the Mechanics of Deformable Bodies, Delft University Press, 1975, pp. 264-281.

B5. Paul, B., "A Review of Rail-Wheel Contact Stress Problems," in Proceedings of Symposium on Railroad Track Mechanics, Pergamon Press, 1978, Ed. by A. Kerr, pp. 323-351.(Based on Report A1).

B6. Paul, B., and Hashemi, J., "An Improved Numerical Method for Counterformal Contact Stress Problems," in Computational Techniques for Interface Problems, AMD-Vol, 30, Ed. by K. C. Park and D. K. Gartlung, American Society of Mechanical Engineers, N.Y., 1978, pp. 165-180. (Same as Report A3).